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L^∞-Bounds of Solutions for Strongly Nonlinear Elliptic Problems with Two Lower Order Terms

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Abstract. In this work, we will prove the existence of bounded solutions in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ for nonlinear elliptic equations $-\operatorname{div}(a(x,u,\nabla u))+g(x,u,\nabla u)+H(x,\nabla u)=f$, where a, g and H are Carathéodory functions which satisfy some conditions, and the right hand side f belongs to $W^{-1,q}(\Omega)$.

Key Words: L^∞-estimate, nonlinear elliptic equations, rearrangement, Sobolev spaces.

AMS Subject Classifications: 35J60, 46E30, 46E35

1 Introduction

Let Ω be a regular bounded domain of $\mathbb{R}^N$, $N>1$ and let us consider the problem:

$$\begin{cases} -\operatorname{div}\big(a(x,u,\nabla u)\big)+g(x,u,\nabla u)+H(x,\nabla u)=f \quad \text{in } \mathcal{D}'(\Omega), \\ u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \end{cases} \tag{1.1}$$

where $-\operatorname{div}(a(x,u,\nabla u))$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ with $p>1$ and $\frac{1}{p}+\frac{1}{p'}=1$, g is a nonlinearity which satisfies the growth condition and also it satisfies a sign condition (i.e., it is an absorption of a lower order term) and H is a reaction term on which suitable hypothesis are made. Moreover the source term f belongs to $W^{-1,q}(\Omega)$ where $q>\frac{N}{p-1}$ and $q\geq p'$.

When $H\equiv 0$, in [3] the authors were interested by the existence of the $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ solutions of $-(a_{ij}u_{x_j})_{x_i}+a_0u=g(x,u,\nabla u)$ with $|g|\leq C_0+b(|u|)|\nabla u|^2$ where a_{ij} is bounded

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measurable, $a_0>0$ and b is a function on $\mathbb{R}^+$, also they were interested by an existence result for $-\triangle_p u+g(x,u,\nabla u)+a_0|u|^{p-1}\text{sign}(u)=f-\text{div}F$ with $|g|\leq C_0+C_1|\nabla u|^p$ where a_0,C_0 and C_1 are strictly positive, f and F are suitably integrable in [5]. For $-(a_{ij}u_{x_j})_{x_i}=g(x,u,\nabla u)-\text{div}f$ with g is satisfying $|g|\leq b+|\nabla u|^p$ for $p=2$, when f and b is suitably integrable, an existence result can be found in [10]. The $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ solution of $-\triangle_p u=g(x,u,\nabla u)$ where g is satisfying $|g|\leq b+|\nabla u|^p$ and b is a suitably integrable study in [6]. In [11] the existence of the $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ solution of $-\text{div}(|\nabla u|^{p-1}\nabla u)=|\nabla u|^p+g-\text{div}f$ with g and f are suitably integrable. Let us point out that more works in this direction can be found in [4, 18]. Recently in [21] when $H\equiv 0$, the authors have proved the existence of bounded solutions of unilateral problems associated with the Dirichlet problems (1.1) in the setting of Orlicz Sobolev space without any restriction on the N-function of the Orlicz spaces, where the function $g(x,u,\nabla u)$ is not satisfying the sign condition.

In the case H is not necessarily the null function, the existence result for the problem (1.1) where $u\in W_0^{1,p}(\Omega)$ was firstly proved in [8] in the case where the functions g does not depend on the gradient and it was secondly proved in [14] using the rearrangement techniques. The existence result of equations with this type with a measure data have been given in [1] and has also been studied in [20] in the case of unilateral problems with L^1-data.

The scope of the present work is to obtain the uniform $L^\infty-$estimates for the solutions of strongly nonlinear elliptic equations (1.1), we based on rearrangement properties [13]. This method has been successfully applied to nonlinear elliptic problems with p-growth in the gradient by Ferone et al. [11]. Such an estimate allows us to prove the existence of a solution of (1.1) see [14]. The smallness conditions on the measure of Ω and some norm of b_1, b, and f are essential in the $L^\infty-$estimates.

Let us briefly summarize the contents of this article. Section 2, contains some preliminary results concerned with the rearrangement propriety. In Section 3, we give the assumption on the data and we show the existence of our result (Theorem 3.1).

2 Preliminary results

We recall here some standard notations and properties which will be used through the paper. Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, and let $\omega:\Omega\to\mathbb{R}$ be a measurable function. If one denotes by $|E|$ the lebesgue measure of a set E, one can define the distribution function $\mu_\omega(t)$ of ω as:

$$\mu_\omega(t)=|\{x\in\Omega:\omega>t\}|,\quad t\geq 0.$$

The decreasing rearrangement ω^* of ω is defined as the generalized inverse function of μ_ω:

$$\omega^*(s)=\inf\{t\geq 0:\mu_\omega(t)\leq s\},\quad s\in[0,|\Omega|].$$