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On the Connection between the Order of Riemann-Liouvile Fractional Calculus and Hausdorff Dimension of a Fractal Function

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Abstract. This paper investigates the fractal dimension of the fractional integrals of a fractal function. It has been proved that there exists some linear connection between the order of Riemann-Liouvile fractional integrals and the Hausdorff dimension of a fractal function.

Key Words: Fractional calculus, Hausdorff dimension, Riemann-Liouvile fractional integral.
AMS Subject Classifications: MR28A80, MR26A33, MR26A30

1 Introduction

Fractional calculus, both of theoretical and practical importance, is an important tool being used to investigate fractal functions and curves. Fractional calculus, such as Riemann-Liouvile fractional integrals, can be effective applied to certain fractals like the Weierstrass function [1]. With the help of the K-dimension, Yao [8,9], Su, and Zhou [11] proved that there exist some linear connection between the order of fractional calculus and the Box dimension, K-dimension, and Packing dimension of graphs of the Weierstrass function. A natural problem is, does this connection still hold for the Hausdorff dimension which is very important in fractal theory? Firstly, we recall the definition of Riemann-Liouvile fractional integral.

Definition 1.1 (see [5])**.** Let f be a function piecewisely continuous on $(0,\infty)$ and integrable on any finite subinterval of $(0,\infty)$. Then we call

$$D^{-v}f(t)=\frac{1}{\Gamma(v)}\int_0^t(t-x)^{v-1}f(x)dx.$$

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Riemann-Liouvile fractional integral of f of order v for $t>0$ and $Re(v)>0$.

This paper considers the Weierstrass function with random phase added to each term, i.e.,

$$f_\Theta(x)=\sum_{n=0}^{\infty}\lambda^{-\alpha n}\sin(2\pi(\lambda^n x+\theta_n)),\quad x\in I, \tag{1.1}$$

where $\lambda>1$, $0<\alpha<1$, $I=[0,1]$, $\Theta=\{\theta_0,\theta_1,\theta_2,\cdots\}$. More details about the type of the Weierstrass function can be found in [1,7].

Definition 1.2. Denote Riemann-Liouvile fractional integral of $\sin(2\pi(\lambda^n x+\theta_n))$ and $\cos(2\pi(\lambda^n x+\theta_n))$ of order v as following

$$\begin{aligned}
S_t(v,\lambda,\theta)&=D^{-v}\sin(\lambda^n x+\theta_n)=\frac{1}{\Gamma(v)}\int_0^t(t-\xi)^{v-1}\sin(2\pi(\lambda^n\xi+\theta_n)),\\
C_t(v,\lambda,\theta)&=D^{-v}\cos(\lambda^n x+\theta_n)=\frac{1}{\Gamma(v)}\int_0^t(t-\xi)^{v-1}\cos(2\pi(\lambda^n\xi+\theta_n)).
\end{aligned}$$

Then define

$$F_\theta(x)=D^{-v}(f_\theta(x))=\sum_{n=0}^{\infty}\lambda^{-\alpha n}S_t(v,\lambda,\theta) \tag{1.2}$$

be R-L fractional integral of $f_\theta(x)$ of order v.

Definition 1.3 (see [2])**.** Let a Borel set $F\in\mathscr{R}^n$ be given as follows. For $s\geq 0$ and $\delta>0$, define

$$\mathscr{H}^s_\delta(F)=\inf\Big\{\sum_{i=1}^{\infty}|U_i|^s:\{U_i\}\text{ is a }\delta\text{-cover of }F\Big\},$$

where $|U|=\sup\{|x-y|:x,y\in U\}$ denotes the diameter of a nonempty set U and the infimum is taken over all countable collections $\{U_i\}$ of sets for which $F\subset\bigcup_{i=1}^{\infty}U_i$ and $0<|U_i|\leq\delta$. As δ decreases, $\mathscr{H}^s_\delta(F)$ cannot decrease, and therefore it has a limit (possibly infinite) as $\delta\to 0$, define

$$\mathscr{H}^s(F)=\lim_{\delta\to 0}\mathscr{H}^s_\delta(F).$$

The quantity $\mathscr{H}^s(F)$ is known as s-dimensional Hausdorff measure of F. For a given F there is a value $\dim_H(F)$ for which $\mathscr{H}^s(F)=\infty$ for $s<\dim_H(F)$ and $\mathscr{H}^s(F)=0$ for $s>\dim_H(F)$. Hausdorff dimension $\dim_H(F)$ is defined to be this value, that is:

$$\dim_H(F)=\inf\{s:\mathscr{H}^s(F)=0\}=\sup\{s:\mathscr{H}^s(F)=\infty\}.$$

For simplicity, let

$$\begin{aligned}
&\widetilde{S}_t(v,\lambda,\theta)=\Gamma(v)S_t(v,\lambda,\theta),\\
&\widetilde{C}_t(v,\lambda,\theta)=\Gamma(v)C_t(v,\lambda,\theta),\\
&C^\alpha(I)=\{f(x):|f(x)-f(y)|\leq c|x-y|^\alpha,\forall x,y\in I\}.
\end{aligned}$$