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Box Dimension of Weyl Fractional Integral of Continuous Functions with Bounded Variation

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Abstract. We know that the Box dimension of $f(x)\in C^1[0,1]$ is 1. In this paper, we prove that the Box dimension of continuous functions with bounded variation is still 1. Furthermore, Box dimension of Weyl fractional integral of above function is also 1.

Key Words: Fractional calculus, box dimension, bounded variation.

AMS Subject Classifications: 28A80, 26A33, 26A30

1 Introduction

As a new branch in mathematics, fractal geometry has proved its value with many applications over many fields. Many initial and conclusive results on fractals were done in [2,3,7]. If $f(x)$ has continuous derivative, it is not difficult to see that Box dimension of $f(x)$ is 1, indeed a regular $1-$set. We want to know whether this result still holds for the function $f(x)$ with bounded variation? What about their fractional integral? Firstly, we give definitions of Hausdorff dimension and Box dimension.

Definition 1.1 (see [1]). Let a Borel set $F\in\mathscr{R}^n$ is as follows. For $s\geq 0$ and $\delta>0$, define

$$\mathscr{H}^s_\delta(F)=\inf\Big\{\sum_{i=1}^{\infty}|U_i|^s:\{U_i\}\text{ is a }\delta\text{-cover of }F\Big\},$$

where $|U|=\sup\{|x-y|:x,y\in U\}$ denotes the diameter of a nonempty set U and the infimum is taken over all countable collections $\{U_i\}$ of sets for which $F\subset\bigcup_i^\infty U_i$ and

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$0<|U_i|\leq\delta$. As δ decreases, $\mathscr{H}^s_\delta(F)$ can not decrease, and therefore it has a limit (possibly infinite) as $\delta\to 0$; define

$$\mathscr{H}^s(F)=\lim_{\delta\to 0}\mathscr{H}^s_\delta(F).$$

The quantity $\mathscr{H}^s(F)$ is known as s-dimensional Hausdorff measure of F. For a given F there is a value $\dim_H(F)$ for which $\mathscr{H}^s(F)=\infty$ for $s<\dim_H(F)$ and $\mathscr{H}^s(F)=0$ for $s>\dim_H(F)$. Hausdorff dimension $\dim_H(F)$ is defined to be this value, that is:

$$dim_H(F)=\inf\{s:\mathscr{H}^s(F)=0\}=\sup\{s:\mathscr{H}^s(F)=\infty\}.$$

Definition 1.2 (see [1])**.** Let F be a any non-empty bounded subset of R^2 and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The lower and upper Box dimensions of F respectively are defined as

$$\underline{\dim}_B(F)=\underline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \tag{1.1}$$

and

$$\overline{\dim}_B(F)=\overline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}. \tag{1.2}$$

If (1.1) and (1.2) are equal, we refer to the common value as the Box dimension of F:

$$\dim_B(F)=\lim_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}.$$

Definition 1.3 (see [6])**.** Let $f(x)$ be a finite function on I, $I=[0,1]$. Let $\{x_i\}_{i=1}^n$ be arbitrary points which satisfy

$$0=x_0<x_1<x_2<\cdots<x_n=1.$$

Write

$$V_f:=\sup_{(x_0,x_1,\cdots,x_n)}\sum_{k=1}^n|f(x_k)-f(x_{k-1})|. \tag{1.3}$$

If (1.3) is finite, then $f(x)$ is of bounded variation on I. Let BV_I denote the set of functions of bounded variation on I. Meanwhile, Let $C(I)$ denote the set of functions which are continuous on I.

Definition 1.4 (see [4])**.** Let $f(x)\in C(I)$ and $0<v<1$. If $f(x)$ is piecewise integrable, we define the Weyl fractional integral of $f(x)$ of order v as

$$D^{-v}f(x)=\frac{1}{\Gamma(v)}\int_x^\infty\frac{f(t)}{(t-x)^{1-v}}dt.$$

In this paper, let $G(f,I)$ denote the graph of $f(x)$ on I, and $\dim_B G(f,I)$ denote the Box dimension of $f(x)$ on I.