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Commutators of Littlewood-Paley Operators on Herz Spaces with Variable Exponent

Hongbin Wang[1,*] and Yihong Wu[2]

[1] *School of Science, Shandong University of Technology, Zibo, Shandong 255049, China*
[2] *Department of Recruitment and Employment, Zibo Normal College, Zibo, Shandong 255130, China*

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Abstract. Let $\Omega \in L^2(\mathrm{S}^{n-1})$ be homogeneous function of degree zero and b be BMO functions. In this paper, we obtain some boundedness of the Littlewood-Paley Operators and their higher-order commutators on Herz spaces with variable exponent.

Key Words: Herz space, variable exponent, commutator, area integral, Littlewood-Paley g_λ^* function.

AMS Subject Classifications: 42B25, 42B35, 46E30

1 Introduction

The theory of function spaces with variable exponent has extensively studied by researchers since the work of Kováčik and Rákosník [7] appeared in 1991. In [9] and [10], the authors proved the boundedness of some Littlewood-Paley operators on variable L^p spaces, respectively.

Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot): E \longrightarrow [1,\infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda > 0$,

$$\int_E \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\Big\{\lambda > 0: \int_E \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx \le 1\Big\}.$$

*Corresponding author. *Email addresses:* `wanghb@sdut.edu.cn` (H. Wang), `wfapple123456@163.com` (Y. Wu)

These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x)=p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L^{p(\cdot)}_{\rm loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\rm loc}(\Omega):=\{f: f\in L^{p(\cdot)}(E)\ \text{for all compact subsets}\ E\subset\Omega\}.$$

Define $\mathcal{P}^0(E)$ to be set of $p(\cdot):E\longrightarrow(0,\infty)$ such that

$$p^-=\operatorname{essinf}\{p(x):x\in E\}>0,\qquad p^+=\operatorname{esssup}\{p(x):x\in E\}<\infty.$$

Define $\mathcal{P}(E)$ to be set of $p(\cdot):E\longrightarrow[1,\infty)$ such that

$$p^-=\operatorname{essinf}\{p(x):x\in E\}>1,\qquad p^+=\operatorname{esssup}\{p(x):x\in E\}<\infty.$$

Denote $p'(x)=p(x)/(p(x)-1)$.

Let $f\in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is defined by

$$Mf(x)=\sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|dy,$$

where $B_r(x)=\{y\in\mathbb{R}^n:|x-y|<r\}$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A\subset\mathbb{R}^n$ by $|A|$ and χ_A respectively. The notation $f\approx g$ means that there exist constants $C_1,C_2>0$ such that $C_1g\le f\le C_2g$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. *If $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and satisfies*

$$|p(x)-p(y)|\le\frac{C}{-\log(|x-y|)},\quad |x-y|\le 1/2, \tag{1.1}$$

and

$$|p(x)-p(y)|\le\frac{C}{\log(|x|+e)},\quad |y|\ge|x|, \tag{1.2}$$

then $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, that is the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 1.2 (see [7])**.** *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. If $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx\le r_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p=1+1/p^--1/p^+.$$