Analysis in Theory and Applications
Anal. Theory Appl., Vol. **32**, No. 1 (2016), pp. 27-37
DOI: 10.4208/ata.2016.v32.n1.3

Some Inequalities for the Polynomial with S-Fold Zeros at the Origin

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Received 8 July 2015; Accepted (in revised version) 27 October 2015

Abstract. Let $p(z)$ be a polynomial of degree n, which has no zeros in $|z|<1$, Dewan et al. [K. K. Dewan and Sunil Hans, Generalization of certain well known polynomial inequalities, J. Math. Anal. Appl., 363 (2010), pp. 38–41] established

$$\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big|\le\frac{n}{2}\Big\{\Big(\Big|\frac{\beta}{2}\Big|+\Big|1+\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|p(z)|-\Big(\Big|1+\frac{\beta}{2}\Big|-\Big|\frac{\beta}{2}\Big|\Big)\min_{|z|=1}|p(z)|\Big\},$$

for any $|\beta|\le 1$ and $|z|=1$. In this paper we improve the above inequality for the polynomial which has no zeros in $|z|<k$, $k\ge 1$, except s-fold zeros at the origin. Our results generalize certain well known polynomial inequalities.

Key Words: Polynomial, s-fold zeros, inequality, maximum modulus, derivative.

AMS Subject Classifications: 30A10, 30C10, 30D15

1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree n, then according to a result known as Bernstein's inequality [3] on the derivative of a polynomial, we have

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{1.1}$$

The result is best possible and equality holds for the polynomials having all its zeros at the origin.

If the polynomial $p(z)$ has all its zeros in $|z|\le 1$, then it was proved by Turan [10] that

$$\max_{|z|=1}|p'(z)|\ge\frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.2}$$

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With equality for those polynomials which have all their zeros at the origin.

For the class of polynomials having no zeros in $|z|<1$, the inequality (1.1) can be replaced by

$$\max_{|z|=1}|p'(z)|\le\frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.3}$$

The inequality (1.3) was conjectured by Erdös and later proved by Lax [6].

As an extension of the inequality (1.2) Malik [7] proved that if $p(z)$ having all its zeros in $|z|\le k$, $k\le 1$, then

$$\max_{|z|=1}|p'(z)|\ge\frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.4}$$

Govil [5] improved the inequality (1.4) and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, then

$$\max_{|z|=1}|p'(z)|\ge\frac{n}{1+k}\Big\{\max_{|z|=1}|p(z)|+\frac{1}{k^{n-1}}\min_{|z|=k}|p(z)|\Big\}. \tag{1.5}$$

As a refinement of the inequality (1.4) Aziz and Zargar [2] proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, with s-fold zeros at the origin, then

$$\max_{|z|=1}|p'(z)|\ge\frac{n+sk}{1+k}\max_{|z|=1}|p(z)|+\frac{n-s}{(1+k)k^s}\min_{|z|=k}|p(z)|. \tag{1.6}$$

Recently Dewan and Hans [4] obtained a refinement of inequalities (1.2) and (1.3). They proved that if $p(z)$ is a polynomial of degree n and has all its zeros in $|z|\le 1$, then for every real or complex number β with $|\beta|\le 1$,

$$\min_{|z|=1}\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big|\ge n\Big|1+\frac{\beta}{2}\Big|\min_{|z|=1}|p(z)|, \tag{1.7}$$

and in the case that $p(z)$ having no zeros in $|z|<1$, they proved that

$$\begin{aligned}&\max_{|z|=1}\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big|\\ \le&\frac{n}{2}\Big\{\Big(\Big|1+\frac{\beta}{2}\Big|+\Big|\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|p(z)|-\Big(\Big|1+\frac{\beta}{2}\Big|-\Big|\frac{\beta}{2}\Big|\Big)\min_{|z|=1}|p(z)|\Big\}.\end{aligned} \tag{1.8}$$

In this paper, we obtain an improvement and generalizations of the above inequalities. For this purpose we first present the following result which is a generalization and refinement of inequalities (1.5), (1.6) and (1.7).

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, with s-fold zeros at the origin where $0\le s\le n$, then for every $\beta\in\mathbb{C}$ with $|\beta|\le 1$ and $|z|=1$,*

$$\Big|zp'(z)+\beta\frac{n+sk}{1+k}p(z)\Big|\ge k^{-n}\Big|n+\beta\frac{n+sk}{1+k}\Big|\min_{|z|=k}|p(z)|. \tag{1.9}$$

With equality for $p(z)=az^n$ where $a\in\mathbb{C}$.