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# On Copositive Approximation in Spaces of Continuous Functions I: The Alternation Property of Copositive Approximation

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**Abstract.** In this paper the author writes a simple characterization for the best copositive approximation to elements of $C(Q)$ by elements of finite dimensional strict Chebyshev subspaces of $C(Q)$ in the case when $Q$ is any compact subset of real numbers. At the end of the paper the author applies this result for different classes of $Q$.

**Key Words**: Strict Chebyshev spaces, best copositive approximation, change of sign.

**AMS Subject Classifications**: 41A65

## 1 Introduction

If $A$ is a subset of a normed linear space $X$, and $x \in X$ then the "distance" $d(x,A)$ from $x$ to $A$ is defined to be; $d(x,A) = \inf\{\|x-y\| : y \in A\}$. If there is $y_0 \in A$ so that $d(x,A) = \|x-y_0\|$ then $y_0$ is called a "best approximation" to $x$ from $A$, and we say that a best approximation for $x$ from $A$ is "attained". The subset $A$ of $X$ is called a Chebyshev subset of $X$ iff each $x \in X$ has a unique best approximation from $A$. If $Q$ is a compact Hausdorff space then $C(Q)$ denotes the Banach space of all continuous real valued functions on $Q$, together with the uniform norm, that is, $\||f\| = \max\{|f(x)| : x \in Q\}$. If $M$ is a subspace of $C(Q), f \in C(Q)$ and $g \in M$, then $g$ is said to be "copositive" with $f$ on $Q$ if $f(x)g(x) \geq 0$ for all $x \in Q$. The element $g_0 \in M$ is called a "best copositive approximation"to $f$ from $M$ iff $g_0$ is copositive with $f$ on $Q$ and; $\|f-g_0\| = \inf\{\|f-g\| : g \in M$, and $g$ is copositive with $f$ on $Q\}$. The set $\{g \in M : g$ is copositive with $f$ on $Q\}$ is closed. So if the dimension of $M$ is finite, then by a simple compactness argument one can show that a best copositive approximation to each $f \in C(Q)$ is attained. If $Q$ is a compact totally ordered space then the $n$-dimensional subspace $M$ of $C(Q)$ is Chebyshev subspace of $C(Q)$ iff each $g \neq 0$ in $M$ has at most $n-1$ zeros, (see Singer [10, Theorem 2.2]). In the case when $Q = [a,b]$, a

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closed and bounded interval of real numbers, the $n$-dimensional Chebyshev subspace $M$ of $C[a,b]$ has the property that each $g \neq 0$ in M has no more than $n-1$ changes of sign, that is, for each $g \neq 0$ in $M$, there do not exist $n+1$ points, $x_1 < x_2 < \cdots < x_{n+1}$ in $Q$ such that $g(x_i)g(x_{i+1}) < 0$ for all $i = 1,2,\cdots,n$. This property turns out to be essential to guarantee the alternation of the error in approximation (see Kamal [4]). Unfortunately this property fails to exist when $Q$ is not connected. This alternation property appeared naturally in the works of different authors. Singer [10], used it under the name of "Chebyshev system". Cheney [1] used it under the name of "Haar condition", and other authors used it when they take $Q$ as a subset of a closed and bounded real interval $[a,b]$ (see Passow, and Taylor [8], Taylor [11] and Zhong [12]). Zielke [13] defines the Chebyshev subspace of $C(Q)$ to be the $n$-dimensional subspace that has this alternating property, and for which each of its element $g \neq 0$ has at most $n-1$.

To avoid ambiguity in this paper, the $n$-dimensional Chebyshev subspace of $C(Q)$ that has this alternating property, will be called a "Strict Chebyshev subspace". So the $n$-dimensional subspace of $C(Q)$ is called a strict Chebyshev subspace, if each element $g \neq 0$ in it has at most $n-1$ zeros and has at most $n-1$ changes of sign. It is clear that when $Q = [a,b]$, a closed and bounded interval of real numbers then each Chebyshev subspace is also a strict Chebyshev subspace.

Copositive approximation has important application as ordinary approximation. In some fields of science (for example Physics), one wants to approximate a complicated function by a simpler one without loosing the sign of the original function. In spite of its importance only few papers appeared studying the copositive approximation by elements of finite dimensional Chebyshev subspaces of $C(Q)$. Some of the early papers are, Schumaker and Taylor [9], and Taylor [11]. Both appeared in 1969, and studied approximation by the so-called "Function with restricted domain". In those papers, the concept of copositive approximation was not mentioned, but you can touch it everywhere. The first time at which the copositive approximation was studied independently, was 1974, by Passow, and Raymon [7]. They devoted the second section of their paper to study the degree of copositive approximation by the subspace of all polynomial of degree less than or equal n, defined on the interval $[-1,1]$. In their paper, they also showed that, in this case, the best copositive approximation is unique. But nothing was mentioned about the characterization of this best copositive approximation. In 1977, Passow and Taylor [8], studied the copositive approximation by elements of a general $n$-dimensional Chebyshev subspace of $C(Q)$, were $Q$ is a closed and bounded real interval $[a,b]$. In their paper, they introduced the concept of "alternant of length $r$", and used it to characterize the best copositive approximation. The definition of "alternant of length $r$"is difficult to follow, and very far from the actual meaning of the word "alternation". The characterization was not complete in this paper because they did not prove the uniqueness of the best copositive approximation. Due to the difficulty of the subject, no papers appeared until 1993. In 1993, Zhong [12] developed the concept of "alternant of length $r$", and produce the concept of "alternates once". He used this concept to characterize the best copositive approximation, and to prove its uniqueness. He worked only on the closed and bounded