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A Note on Padé Approximants Pairs as Limits of Algebraic Polynomials Pairs of Weighted Best Approximation in Orlicz Spaces

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Abstract. In this short note, we show the behavior in Orlicz spaces of best approximations by algebraic polynomials pairs on union of neighborhoods, when the measure of them tends to zero.

Key Words: Best approximation pair, Padé approximant pair, Orlicz spaces.

AMS Subject Classifications: 41A30, 41A21

1 Introduction

Let $\emptyset \neq X \subset \mathbb{R}$ be an open and bounded set. We denote by $\mathcal{M} = \mathcal{M}(X)$ the equivalence class of all real Lebesgue measurable functions on X. Let Φ be the set of convex functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\phi(x) > 0$ for $x > 0$ and $\phi(0) = 0$.

For each $\phi \in \Phi$, define

$$L^\phi = L^\phi(X) = \Big\{ f \in \mathcal{M} \colon \int_X \phi(\alpha|f(x)|)dx < \infty \ \text{ for some } \ \alpha > 0 \Big\}.$$

The space L^ϕ is called an Orlicz space determined by ϕ. This space is endowed with the Luxemburg norm defined by

$$\|f\|_\phi = \inf\Big\{ \lambda > 0 \colon \int_X \phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \leq 1 \Big\},$$

and so it becomes a Banach space. Sometimes we write $\|\cdot\|_{L^\phi(W)}$ instead of $\|f\chi_W\|_\phi$, where χ_W denotes the characteristic function on $W \subset X$.

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We assume that $\phi \in \Phi$ satisfies the Δ_2-condition, that is, there exists a constant $\gamma > 0$ such that $\phi(2x) \le \gamma\phi(x)$ for all $x \ge 0$. In this case,

$$\int_X \phi\Big(\frac{|f(x)|}{\|f\|_\phi}\Big)dx = 1.$$

A detailed treatment about these subjects may be found in [5].

Given $x_1 < \cdots < x_k$ in X, $k \ge 1$, for $\delta > 0$ small enough we define a net of pairwise disjoint sets $V_j = V_j(\delta) := x_j + \varepsilon_j(\delta) A_j \subset X$, $1 \le j \le k$, where $\varepsilon_j = \varepsilon_j(\delta) \searrow 0$ as $\delta \to 0$, and each interval A_j, independent of δ, has Lebesgue measure 1.

Let $a \in \mathbb{R}$, $n,m \in \mathbb{N} \cup \{0\}$ and let Π^n be the class of algebraic polynomials with real coefficients of degree at most n. For $r \in \{0,1\}$, let $\Pi^m(a,r) = \{Q \in \Pi^m : Q(a) = r\}$ and we consider the sets

$$\mathcal{S}^n_m(a) := \Pi^n \times \Pi^m(a,1) \quad \text{and} \quad \Pi^n_m(a) := \Big\{\frac{P}{Q} : (P,Q) \in \Pi^n \times \Pi^m(a,0),\ Q \neq 0\Big\}.$$

Given a function $f \in L^\phi$, we say that $(P_\delta, Q_\delta) \in \mathcal{S}^n_m(a)$ is a *best* $\|\cdot\|_\phi$*-approximant pair* of f from $\mathcal{S}^n_m(a)$ respect to $\|\cdot\|_{L^\phi(V)}$ if

$$\|fQ_\delta - P_\delta\|_{L^\phi(V)} \le \|fQ - P\|_{L^\phi(V)}, \quad (P,Q) \in \mathcal{S}^n_m(a), \tag{1.1}$$

where $V = \bigcup_{j=1}^k V_j$. It is easy to see that (P_δ, Q_δ) exists. In fact, let $Q_*(x) = 1$, $x \in \mathbb{R}$. Then $\Pi^m(a,1) = Q_* + \Pi^m(a,0)$ and we see that existence of a minimizing pair for (1.1) is equivalent to the existence of a minimum of

$$\|f - R\|_{L^\phi(V)}, \quad R \in \mathcal{R}^n_m(f,a), \tag{1.2}$$

where

$$\mathcal{R}^n_m(f,a) := f\Pi^m(a,0) + \Pi^n$$

is a finite dimensional subspace of L^ϕ. Clearly, (1.2) is minimized by some $R_0 = fQ_0 + P_0 \in \mathcal{R}^n_m(f,a)$, so that $(P_0, Q_* - Q_0)$ is a best $\|\cdot\|_\phi$-approximant pair of f from $\mathcal{S}^n_m(a)$ respect to $\|\cdot\|_{L^\phi(V)}$.

We observe that if $f \notin \Pi^n_m(a)$, then $\mathcal{R}^m_n(f,a)$ has dimension $n+m+1$ and $\mathcal{R}^n_m(f,a) = f\Pi^m(a,0) \oplus \Pi^n$

If the net (P_δ, Q_δ) has a limit in $\mathcal{S}^n_m(a)$ as $\delta \to 0$, this limit is called a best local $\|\cdot\|_\phi$-approximation of type (n,m) of f from $\mathcal{S}^n_m(a)$ on $\{x_1, \cdots, x_k\}$.

We denote by $PC^t(X)$ the class of functions with $t-1$ continuous derivatives and bounded, piecewise continuous t^{th} derivative on X. Let $f \in PC^t(X)$, $(P,Q) \in \Pi^n \times \Pi^m$, and let $\langle i_j \rangle$ be an ordered N-tuple of nonnegative integers with $i_j \le t$ and $\sum_{j=1}^N i_j = n+m+1$. If

$$(fQ-P)^{(i)}(y_j) = 0, \quad j = 1,2,\cdots,N, \quad i = 0,1,\cdots,i_j - 1, \tag{1.3}$$