Analysis in Theory and Applications
Anal. Theory Appl., Vol. **31**, No. 1 (2015), pp. 81-91
DOI: 10.4208/ata.2015.v31.n1.7

Some Integral Mean Estimates for Polynomials with Restricted Zeros

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Received 19 January 2014; Accepted (in revised version) 11 March 2015

Abstract. Let $P(z)$ be a polynomial of degree n having all its zeros in $|z|\le k$. For $k=1$, it is known that for each $r>0$ and $|\alpha|\ge 1$,

$$n(|\alpha|-1)\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}}\le\Big\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\Big\}^{\frac{1}{r}}\max_{|z|=1}|D_\alpha P(z)|.$$

In this paper, we shall first consider the case when $k\ge 1$ and present certain generalizations of this inequality. Also for $k\le 1$, we shall prove an interesting result for Lacunary type of polynomials from which many results can be easily deduced.

Key Words: Polynomial, zeros, polar derivative.

AMS Subject Classifications: 30A10, 30C10, 30D15

1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. It was shown by Turan [21] that if $P(z)$ has all its zeros in $|z|\le 1$, then

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.1}$$

More generally, if the polynomial $P(z)$ has all its zeros in $|z|\le k\le 1$, it was proved by Malik [12] that the inequality (1.1) can be replaced by

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{1+k}\max_{|z|=1}|P(z)|, \tag{1.2}$$

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while as Govil [6] proved that if all the zeros of $P(z)$ lie in $|z|\le k$ where $k\ge 1$, then

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{1+k^n}\max_{|z|=1}|P(z)|. \tag{1.3}$$

As an improvement of (1.3), Govil [7] proved that if $P(z)$ has all its zeros in $|z|\le k$ where $k\ge 1$, then

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{1+k^n}\Big(\max_{|z|=1}|P(z)|+\min_{|z|=k}|P(z)|\Big). \tag{1.4}$$

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\Big\{\frac{D_\alpha P(z)}{\alpha}\Big\}=P'(z). \tag{1.5}$$

Shah [18] extended (1.1) to the polar derivative of $P(z)$ and proved that if all the zeros of the polynomial $P(z)$ lie in $|z|\le 1$, then

$$\max_{|z|=1}\big|D_\alpha P(z)\big|\ge\frac{n}{2}\big(|\alpha|-1\big)\max_{|z|=1}|P(z)|,\quad |\alpha|\ge 1. \tag{1.6}$$

Aziz and Rather [3] generalised (1.6) which also extends (1.2) to the polar derivative of a polynomial. In fact, they proved that if all the zeros of $P(z)$ lie in $|z|\le k$ where $k\le 1$, then for every real or complex number α with $|\alpha|\ge k$,

$$\max_{|z|=1}\big|D_\alpha P(z)\big|\ge n\Big(\frac{|\alpha|-k}{1+k}\Big)\max_{|z|=1}|P(z)|. \tag{1.7}$$

Further as a generalization of (1.3) to the polar derivative of a polynomial, Aziz and Rather [3] proved that if all the zeros of $P(z)$ lie in $|z|\le k$ where $k\ge 1$, then for every real or complex number α with $|\alpha|\ge k$,

$$\max_{|z|=1}\big|D_\alpha P(z)\big|\ge n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\max_{|z|=1}|P(z)|. \tag{1.8}$$

Recently Govil and McTume [8] sharpened (1.8) and proved that if all the zeros of $P(z)$ lie in $|z|\le k$, $k\ge 1$, then for every real or complex number α with $|\alpha|\ge 1+k+k^n$,

$$\begin{aligned}\max_{|z|=1}\big|D_\alpha P(z)\big|\ge{}& n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\max_{|z|=1}|P(z)|\\&+n\Big(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\Big)\min_{|z|=k}|P(z)|.\end{aligned} \tag{1.9}$$