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Construction Theory of Function on Local Fields

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Abstract. We establish the construction theory of function based upon a local field K_p as underlying space. By virture of the concept of pseudo-differential operator, we introduce "fractal calculus" (or, p-type calculus, or, Gibbs-Butzer calculus). Then, show the Jackson direct approximation theorems, Bermstein inverse approximation theorems and the equivalent approximation theorems for compact group $D(\subset K_p)$ and locally compact group $K_p^+(=K_p)$, so that the foundation of construction theory of function on local fields is established. Moreover, the Jackson type, Bernstein type, and equivalent approximation theorems on the Hölder-type space $C^\sigma(K_p)$, $\sigma>0$, are proved; then the equivalent approximation theorem on Sobolev-type space $W_\sigma^r(K_p)$, $\sigma\geq 0$, $1\leq r<+\infty$, is shown.

Key Words: Construction theory of function, local field, fractal calculus, approximation theorem, Hölder-type space.

AMS Subject Classifications: 41A65, 28A20

1 Concept and notation

A local field K_p is a locally compact, non-trivial, totally disconnected, non-Archimedean norm valued, T_2-type, complete topological field [1]. It can be a p-series field, or its finite algebraic extension field (with addition $+$, multiplication $\times$, term by term, mod p, and no carrying); or can be a p-adic filed, or its finite algebraic extension field (with $+$, $\times$, term by term, mod p, carrying from left to right), with $p\geq 2$ prime.

This kind of fields has important theoretical and applied meaning, for example, the dyadic system in the computer science, and switch functions in physics science, they are special cases of local fields at $p=2$.

We concern the cases of p-series field and p-adic field, denoted by $K_p\equiv(K_p,+,\times)$, and call them local fields. For the algebraic extension of K_p, denoted by K_q, $q=p^c$, $c\in\mathbb{N}$, we refer to [1].

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1.1 Haar measure and Haar integral on a local field K_p

The addition group $K_p^+ \equiv K_p$ of a local field $K_p \equiv (K_p,+,\times)$ is a locally compact group, there exist the Haar measure and Haar integral with invariance of translation. The Haar measure of a Haar measurable subset $A \subset K_p$ is denoted by $|A|$; the Haar integral of a Haar measurable function $f: K_p \to \mathbb{C}$ is denoted by $\int_{K_p} f(x)dx$.

1.2 Non-Archimedean valued norm $|x|$ on K_p

If a mapping $|x|: K_p \to [0,+\infty)$ satisfies: (1) $|x| \geq 0$, $|x|=0 \Leftrightarrow x=0$; (2) $|x\times y|=|x||y|$; (3) $|x+y| \leq \max\{|x|,|y|\}$; Then $|x|$ is said to be a non-Archimedean valued norm of $x \in K_p$.

There exists an element $\beta \in K_p$ with $|\beta|=p^{-1}$ in K_p, called prime element. $\forall x \in K_p$ can be expressed as

$$x=x_{-l}\beta^{-l}+x_{-l+1}\beta^{-l+1}+\cdots+x_{-1}\beta^{-1}+x_0\beta^0+x_1\beta^1\cdots, \tag{1.1}$$

where $x_j \in \{0,1,\cdots,p-1\}$, $j=-l,-l+1,\cdots$, $l \in \mathbb{Z}$.

Each $x \in K_p$ in p-series field or in p-aidc field can be expressed as the form in (1.1), the difference is: the operations in p-series field are term by term, mod p, no carrying; whereas, the operations in p-aidc field are term by term, mod p, carrying from left to right.

The range of non-Archimedean valued norm is $|x| \in \{p^{-k}: k \in \mathbb{Z}\}$.

1.3 Important subsets in K_p

(1) Compact group in K_p (ring of integers): $D=\{x\in K_p: |x|\leq 1\}$, it is a unique maximal compact subring in K_p, and is an open, closed, compact subset with Haar measure $|D|=1$.

(2) Unit open ball in K_p (prime ideal): $B=\{x\in K_p: |x|<1\}$, it is a unique maximal ideal in D, also principle ideal, prime ideal; and is open, closed, compact subset with Haar measure $|B|=p^{-1}$.

(3) Ball in K_p (fractional ideal): $B^k=\{x\in K_p: |x|\leq p^{-k}\}$, $k\in\mathbb{Z}$, it is a ball in K_p with center $0\in K_p$ and radius p^{-k}; and is open, closed, compact subset with Haar measure $|B^k|=p^{-k}$, $k\in\mathbb{Z}$.

(4) Base for neighborhood system of $0\in K_p: \{B^k \subset K_p: k\in\mathbb{Z}\}$ satisfies $B^{k+1}\subset B^k$, $k\in\mathbb{Z}$; $K_p=\bigcup_{k=-\infty}^{+\infty} B^k$, $\{0\}=\bigcap_{k=-\infty}^{+\infty} B^k$; the set of all p-coset representatives of B^1 in D is $D/B^1=\{0\times\beta^0+B^1,\ 1\times\beta^0+B,\cdots,(p-1)\times\beta^0+B^1\}$, it is isomorphic with the finite Galois field $D/B^1 \xleftrightarrow{iso.} GF(p)$.

(5) Character group of $K_p: \Gamma_p=\{\chi: K_p\to\mathbb{C},\ \chi(x_1+x_2)=\chi(x_1)\chi(x_2);\ |\chi(x)|=1\}$ is the character group of K_p, it is a locally compact group, and $\Gamma_p \xleftrightarrow{iso.} K_p$.