Analysis in Theory and Applications
Anal. Theory Appl., Vol. **31**, No. 1 (2015), pp. 1-12

DOI: 10.4208/ata.2015.v31.n1.1

The Strong Approximation of Functions by Fourier-Vilenkin Series in Uniform and Hölder Metrics

T. V. Iofina and S. S. Volosivets*

Saratov State University, Faculty of Mathematics and Mechanics, Astrakhanskaya str., 83, 410012, Saratov, Russia

Received 15 May 2012; Accepted (in revised version) 15 January 2015

Abstract. We will study the strong approximation by Fourier-Vilenkin series using matrices with some general monotone condition. The strong Vallee-Poussin, which means of Fourier-Vilenkin series are also investigated.

Key Words: Vilenkin systems, strong approximation, generalized monotonicity.

AMS Subject Classifications: 40F05, 42C10, 43A55, 43A75

1 Introduction

Let $\mathbf{P}=\{p_n\}_{n=1}^{\infty}$ be a sequence of natural numbers such that $2\le p_i\le N$, $i\in\mathbb{N}=\{1,2,\cdots\}$. By definition $\mathbb{Z}(p_j)=\{0,1,\cdots,p_j-1\}$, $m_0=1$, $m_n=p_1p_2\cdots p_n$ for $n\in\mathbb{N}$. Then every $x\in[0,1)$ has an expansion

$$x=\sum_{n=1}^{\infty}\frac{x_n}{m_n},\quad x_n\in\mathbb{Z}(p_n),\quad n\in\mathbb{N}. \tag{1.1}$$

For $x=k/m_l$, $0<k<m_l$, $k,l\in\mathbb{N}$, we take the expansion with a finite number of $x_n\neq 0$. Let $G(\mathbf{P})$ be the Abel group of sequences $\mathbf{x}=(x_1,x_2,\cdots)$, $x_n\in\mathbb{Z}(p_n)$, with addition $\mathbf{x}\oplus\mathbf{y}=\mathbf{z}=(z_1,z_2,\cdots)$, where $z_n\in\mathbb{Z}(p_n)$ and $z_n=x_n+y_n \pmod{p_n}$, $n\in\mathbb{N}$. We define maps $g:[0,1)\to G(\mathbf{P})$ and $\lambda:G(\mathbf{P})\to[0,1)$ by formulas $g(x)=(x_1,x_2,\cdots)$, where x is in the form (1.1) and $\lambda(\mathbf{x})=\sum_{i=1}^{\infty}x_i/m_i$, where $\mathbf{x}\in G(\mathbf{P})$. Then for $x,y\in[0,1)$, we can introduce $x\oplus y:=\lambda(g(x)\oplus g(y))$, if $\mathbf{z}=g(x)\oplus g(y)$ does not satisfy equality $z_i=p_i-1$ for all $i\ge i_0$. In a similar way, we introduce $x\ominus y$ and for all $x,y\in[0,1)$ generalized distance

*Corresponding author. *Email addresses:* `IofinaT@mail.ru` (T. V. Iofina), `VolosivetsSS@mail.ru` (S. S. Volosivets)

$\rho(x,y)=\lambda(g(x)\ominus g(y))$. Every $k\in\mathbb{Z}_+=\{0,1,2,\cdots\}$ can be expressed uniquely in the form of

$$k=\sum_{n=1}^{\infty}k_n m_{n-1},\quad k_n\in\mathbb{Z}_n,\quad n\in\mathbb{N}. \tag{1.2}$$

For a given $x\in[0,1)$ with expansion (1.1) and $k\in\mathbb{Z}_+$ with expansion (1.2), we set $\chi_k(x)=\exp(2\pi i\sum_{j=1}^{\infty}x_jk_j/p_j)$. The system $\{\chi_k\}_{k=0}^{\infty}$ is called a multiplicative or Vilenkin system. It is orthonormal and complete in $L[0,1)$ and we have

$$\chi_k(x\oplus y)=\chi_k(x)\chi_k(y),\quad \chi_k(x\ominus y)=\chi_k(x)\overline{\chi_k(y)},$$

for a.e. y, whenever $x\in[0,1)$ is fixed (see [8, Section 1.5]).

The Fourier-Vilenkin coefficients and partial Fourier-Vilenkin sums for $f\in L^1[0,1)$ are defined by

$$\hat{f}(k)=\int_0^1 f(x)\overline{\chi_k(x)}dx,\quad k\in\mathbb{Z}_+,\quad S_n(f)(x)=\sum_{k=0}^{n-1}\hat{f}(k)\chi_k(x),\quad n\in\mathbb{N}.$$

If $f,g\in L^1[0,1)$, then $f*g(x)=\int_0^1 f(x\ominus t)g(t)dt=\int_0^1 f(t)g(x\ominus t)dt$. For Dirichlet kernel $D_n(t)=\sum_{k=0}^{n-1}\chi_k(t)$, $n\in\mathbb{N}$, we have an equality $S_n(f)(x)=\int_0^1 f(x\ominus t)D_n(t)dt$. The space $L^p[0,1)$, $1\le p<\infty$ consists of all measurable functions f on $[0,1)$ with finite norm $\|f\|_p=(\int_0^1|f(t)|^p dt)^{1/p}$. If $\omega^*(f,\delta)_\infty:=\sup\{|f(x)-f(y)|:x,y\in[0,1),\ \rho(x,y)<\delta\}$, $\delta\in[0,1]$, then $C^*[0,1)$ contains all functions f with property $\lim_{h\to 0}\omega^*(f,h)_\infty=0$ and finite norm $\|f\|_\infty=\sup_{x\in[0,1)}|f(x)|$.

Let us introduce a modulus of continuity $\omega^*(f,\delta)_p=\sup_{0<h<\delta}\|f(x\ominus h)-f(x)\|_p$ in $L^p[0,1)$, $1\le p<\infty$. If $\mathcal{P}_n=\{f\in L^1[0,1):\hat{f}(k)=0,k\ge n\}$, then $E_n(f)_p=\inf\{\|f-t_n\|_p,t_n\in\mathcal{P}_n\}$, $1\le p\le\infty$. Let $\omega(\delta)$ be a function of modulus of continuity type ($\omega(\delta)\in\Omega$), i.e., $\omega(\delta)$ is continuous and increasing on $[0,1)$ and $\omega(0)=0$. Then the space $H_p^\omega[0,1)$ consists of $f\in L^p[0,1)$ $(1\le p<\infty)$ or $f\in C^*[0,1)$ $(p=\infty)$ such that $\omega^*(f,\delta)_p\le C\omega(\delta)$, where C depends only on f. Denote by h_p^ω the subspace of H_p^ω consioting of all functions f such that $\lim_{h\to 0}\omega^*(f,h)_p/\omega(h)=0$. The spaces $h_p^\omega[0,1)$ and $H_p^\omega[0,1)$, $1\le p\le\infty$, with the norm $\|f\|_{p,\omega}=\|f\|_p+\sup_{0<h<1}\omega^*(f,h)_p/\omega(h)$ are Banach ones. In $h_p^\omega[0,1)$ we can consider $E_n(f)_{p,\omega}=\inf\{\|f-t_n\|_{p,\omega},t_n\in\mathcal{P}_n\}$, $n\in\mathbb{N}$.

Let $A=\{a_{nk}\}_{n,k=1}^{\infty}$ be a lower triangle matrix such that

$$a_{n,k}\ge 0,\quad n,k\in\mathbb{N},\quad \sum_{k=1}^{n}a_{n,k}=1. \tag{1.3}$$

Using matrix A, we can define a summation method by formula

$$T_n(f)(x)=\sum_{k=1}^{n}a_{n,k}S_k(f)(x).$$