Analysis in Theory and Applications
Anal. Theory Appl., Vol. **31**, No. 1 (2015), pp. 13-24

DOI: 10.4208/ata.2015.v31.n1.2

Parameterized Littlewood-Paley Operators and Their Commutators on Lebesgue Spaces with Variable Exponent

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Received 23 May 2014; Accepted (in revised version) 22 January 2015

Abstract. In this paper, by applying the technique of the sharp maximal function and the equivalent representation of the norm in the Lebesgue spaces with variable exponent, the boundedness of the parameterized Littlewood-Paley operators, including the parameterized Lusin area integrals and the parameterized Littlewood-Paley g_λ^*-functions, is established on the Lebesgue spaces with variable exponent. Furthermore, the boundedness of their commutators generated respectively by BMO functions and Lipschitz functions are also obtained.

Key Words: Parameterized Littlewood-Paley operators, commutators, Lebesgue spaces with variable exponent.

AMS Subject Classifications: 42B20, 42B25, 42B35

1 Introduction and main results

The Littlewood-Paley operators, including Lusin area integrals, Littlewood-Paley g-functions and g_λ^*-functions, play very important roles in harmonic analysis and PDE (see [1–4]). In [5], Lu and Yang investigated the behavior of Littlewood-Paley operators in the space $CBMO_p(\mathbb{R}^n)$. In 2009, Xue and Ding gave weighted estimates for Littlewood-Paley operators and their commutators (see [6]). In 2013, Wei and Tao proved Litttleood-Paley operators with rough kernels are bounded on weighted $(L^q,L^p)^\alpha(\mathbb{R}^n)$ spaces (see [7]).

In 1960, the parameterized Littlewood-Paley operators were discussed by Hörmander (see [8]) for the first time. Now, let us review the definitions of the parameterized Lusin area integral and the parameterized Littlewood-Paley g_λ^*-function.

Let S^{n-1} denote the unit sphere of $\mathbb{R}^n$ equipped with Lebesgue measure $d\sigma(x')$ and $\psi^\rho(x)=\Omega(x)|x|^{-n+\rho}\chi_{\{|x|\le 1\}}$, where $0<\rho<n$ and Ω satisfies the following conditions:

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(a) $\Omega(\lambda x)=\Omega(x)$ for all $\lambda>0$;

(b) $\int_{\mathbb{R}^n}\Omega(x')d\sigma(x')=0$;

(c) $\Omega\in L^1(S^{n-1})$.

Then the parameterized Lusin area integral S^ρ and the parameterized Littlewood-Paley g_λ^*-function $g_\lambda^{*,\rho}$ are defined respectively by

$$S^\rho(f)(x)=\Big(\iint_{\Gamma_a(x)}|\psi_t^\rho*f(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}$$

and

$$g_\lambda^{*,\rho}(f)(x)=\Big(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}|\psi_t^\rho*f(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\Gamma(x)=\{(t,y)\in\mathbb{R}_+^{n+1}:|y-x|<t\}$, $\lambda>1$.

In [9], Torchinsky and Wang studied the boundedness of the operators S^ρ and $g_\lambda^{*,\rho}$ on weighted $L^2(\mathbb{R}^n)$ for $\rho=1$ and $\Omega(x)\in Lip_\alpha(S^{n-1})$ $(0<\alpha\le 1)$. For general ρ, Sakamoto and Yabuta considered the L^p boundedness of S^ρ and $g_\lambda^{*,\rho}$ in [10]; Wei and Tao given the boundedness of parameterized Litttlewood-Paley operators with rough kernels on weighted weak Hardy spaces in [11].

Now let us turn to the introduction of the corresponding m-order commutators of the parameterized Littlewood-Paley operators above. Let $b\in L^1_{\rm loc}(\mathbb{R}^n)$, $m\in\mathbb{N}$, the commutators $[b^m,S^\rho]$ and $[b^m,g_\lambda^{*,\rho}]$ are defined respectively by

$$[b^m,S^\rho](f)(x)=\Big(\iint_{\Gamma_a(x)}\Big|\frac{1}{t^\rho}\int_{|y-x|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac12}$$

and

$$\begin{aligned}&[b^m,g_\lambda^{*,\rho}](f)(x)\\=&\Big(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\frac{1}{t^\rho}\int_{|y-x|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac12}.\end{aligned}$$

In 2007, Ding and Xue established the weak $L\log L$ estimates of the commutators $[b^m,S^\rho]$ and $[b^m,g_\lambda^{*,\rho}]$ for $b\in BMO(\mathbb{R}^n)$ (see [12]). In 2009, Chen and Ding investigated the characterization of the commutators for the parameterized Littlewood-Paley operators (see [13,14]).

On the other hand, Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one of the important class function spaces due to the seminal paper [15] by Kováčik and Rákosník. In the past twenty years, the theory of these spaces was made progress rapidly, and the study of which was widely applied in some fields such as fluid dynamics, elasticity dynamics, calculus of variations and differential equations with non-standard growth conditions (see [16–20]). In [21], Cruz-Uribe, Fiorenza, Martell and Pérez studied the extrapolation theorem which leads the boundedness of some classical operators