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L^q Inequalities and Operator Preserving Inequalities

M. Bidkham* and S. Ahmadi

Department of Mathematics, Semnan University, Semnan, Iran

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Abstract. Let $\mathbb{P}_n$ be the class of polynomials of degree at most n. Rather and Shah [15] proved that if $P\in\mathbb{P}_n$ and $P(z)\neq 0$ in $|z|<1$, then for every $R>0$ and $0\leq q<\infty$,

$$|B[P(Rz)]|_q\leq\frac{|R^nB[z^n]+\lambda_0|_q}{|1+z^n|_q}|P(z)|_q,$$

where B is a B_n-operator.

In this paper, we prove some generalization of this result which in particular yields some known polynomial inequalities as special. We also consider an operator D_α which maps a polynomial $P(z)$ into $D_\alpha P(z):=nP(z)+(\alpha-z)P'(z)$ and obtain extensions and generalizations of a number of well-known L_q inequalities

Key Words: Complex polynomial, polar derivative, B-operator.

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1 Introduction and statement of results

Let $\mathbb{P}_n$ be the class of polynomials $P(z)=\sum_{j=0}^n a_jz^j$ of degree at most n. For $P\in\mathbb{P}_n$, define

$$\|P(z)\|_q:=\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^q\right\}^{\frac{1}{q}},\quad 1\leq q<\infty,$$

and

$$\|P(z)\|_\infty:=\max_{|z|=1}|P(z)|.$$

If $P\in\mathbb{P}_n$, then

$$\|P'(z)\|_q\leq n\|P(z)\|_q,\quad q\geq 1,\tag{1.1}$$

and

$$\|P(Rz)\|_q\leq R^n\|P(z)\|_q,\quad R>1,\quad q>0.\tag{1.2}$$

*Corresponding author. *Email addresses:* `mdbidkham@gmail.com` (M. Bidkham), `sa_ahmadi58@yahoo.com` (S. Ahmadi)

The inequality (1.1) is found by Zygmund [17], while the inequality (1.2) is a simple deduction from the maximum modulus principle [8]. Arestov [1] proved that (1.1) remains true for $0<q<1$ as well. If we restrict ourselves to the class of polynomials having no zeros in $|z|<K$, $K\geq 1$, the inequality (1.1) can be improved. In fact, it was shown by Malik [9] that

$$\|P'(z)\|_\infty \leq \frac{n}{1+K}\|P(z)\|_\infty. \tag{1.3}$$

Govil and Rahman [7] extended (1.3) to L^q inequality and proved that if $P(z)$ does not vanish in $|z|<K$, $K\geq 1$, then

$$\|P'(z)\|_q \leq \frac{n}{\|K+z\|_q}\|P(z)\|_q, \quad q\geq 1, \tag{1.4}$$

which contains the inequality (1.3) as a special case and Gardner and Weems [6] extended it for $0<q<1$.

Also Boas and Rahman [4] proved for $q\geq 1$ and Rahman and Schmeisser [11] extended it for $0<q<1$ that if $P(z)\neq 0$ for $|z|<1$, then

$$\|P(Rz)\|_q \leq \frac{\|R^n z+1\|_q}{\|1+z\|_q}\|P(z)\|_q, \quad R>1, \quad q>0. \tag{1.5}$$

Rahman [15] introduced operators preserving inequalities between polynomials.

Let T be a linear operator carrying polynomials in $\mathbb{P}_n$ into polynomials in $\mathbb{P}_n$. T is a B_n-operator if for every polynomial $P(z)$ of degree n having all its zeros in the closed unit disc, $T[P]$ has all its zeros in the closed unit disc.

Let λ_0, λ_1 and λ_2 be such that $\lambda_0+n\lambda_1 z+n(n-1)\lambda_2 z^2\neq 0$ for $Re(z)>n/4$. Then, the operator B, which associates with a polynomial $P(z)$ of degree at most n the polynomial

$$B[P(z)]=\lambda_0 P(z)+\frac{1}{1!}\lambda_1\Big(\frac{n}{2}z\Big)P'(z)+\frac{1}{2!}\lambda_2\Big(\frac{n}{2}z\Big)^2 P''(z), \tag{1.6}$$

is a B_n-operator [15].

Rahman [12] proved that

$$|B[P(z)]|\leq M|B[z^n]|, \quad |z|\geq 1,$$

where $|P(z)|\leq M$ for $|z|=1$.

For the class of polynomial having no zeros in $|z|<1$, Rather and Shah [15] proved the following result:

Theorem 1.1. *Let $P\in\mathbb{P}_n$ and $P(z)\neq 0$ in $|z|<1$, then for every $R>1$ and $0\leq q<\infty$,*

$$\|B[P(Rz)]\|_q \leq \frac{\|R^n B[z^n]+\lambda_0\|_q}{\|1+z^n\|_q}\|P(z)\|_q, \tag{1.7}$$

where B is given by (1.6). *The result is sharp, as is shown by the extremal polynomial $P(z)=az^n+b$, $|a|=|b|\neq 0$.*