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The Boundedness of the Commutator for Riesz Potential Associated with Schrödinger Operator on Morrey Spaces

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Abstract. Let $\mathcal{L}=-\Delta+V$ be the Schrödinger operator on $\mathbb{R}^d$, where Δ is the Laplacian on $\mathbb{R}^d$ and $V\neq 0$ is a nonnegative function satisfying the reverse Hölder's inequality. The authors prove that Riesz potential $\mathcal{I}_\beta$ and its commutator $[b,\mathcal{I}_\beta]$ associated with $\mathcal{L}$ map from $M_{\alpha,v}^{p,q}$ into $M_{\alpha,v}^{p_1,q_1}$.

Key Words: Reverse Hölder class, commutator, Schrödinger operator.

AMS Subject Classifications: 42B25, 42B35, 26D15

1 Introduction

In this paper,we consider the Schrödinger differential operator

$$\mathcal{L}=-\Delta+V$$

in $\mathbb{R}^n$, $n\geq 3$.The function V is non-negative, $V\neq 0$, and belongs to a reverse Hölder class RH_q for some exponent $q>n/2$, i.e., there exists a constant C such that

$$\Big(\frac{1}{|B|}\int_B V(y)dy\Big)^{\frac{1}{q}}\leq\frac{C}{|B|}\int_B V(y)dy,$$

for every ball $B\subset\mathbb{R}^n$.

The authors in [1] extended the classe of BMO functions to the new class $BMO_{\mathcal{L}}^{\theta}$ with $\theta>0$. According to [1],the new BMO space $BMO_{\mathcal{L}}^{\theta}$ with $\theta>0$ is defined as a set of all locally integrable functions b satisfying

$$\frac{C}{|B|}\int_B|b(y)-b_B|dy\leq\Big(1+\frac{r}{\rho(x)}\Big)^{\theta},$$

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where $B=B(x,r)$, and $b_B=\frac{1}{|B|}\int_B b(y)dy$. The norm for $b\in BMO_{\mathcal{L}}^{\theta}$ is denoted by $\|b\|_\theta$. Clearly $BMO_{\mathcal{L}}^{\theta_1}\subset BMO_{\mathcal{L}}^{\theta_2}$ for $\theta_1\le\theta_2$ and $BMO_{\mathcal{L}}^{0}=BMO$.

Let $p\in[1,\infty)$, $\alpha\in(-\infty,\infty)$ and $\lambda\in[0,n)$, for $f\in L_{loc}^p(\mathbb{R}^n)$ and $V\in RH_q$ $(q>1)$, we say $f\in M_{\alpha,v}^{p,q}(\mathbb{R}^n)$ (Morrey spaces related to the nonnegative potential V) provided that

$$\|f\|_{M_{\alpha,v}^{p,q}(\mathbb{R}^n)}=\sup_{B(x_0,r)\subset\mathbb{R}^n}\Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha}|B|^{\frac{1}{q}-\frac{1}{p}}\Big(\int_{B(x_0,r)}|f(x)|^p\Big)^{\frac{1}{p}}<\infty,$$

where $B=B(x_0,r)$ denotes a ball with centered at x_0 and radius r.

It is well known that the boundedness of the standard Calderón-Zygmund operators and their commutators have been established on the class of Morrey spaces [3]. Hence,it will be an interesting question whether we can establish the boundedness of fractional integrals related to Schrödinger operators on Morrey spaces with nonegative potentials. Let's give the definition of the fractional integral associated with $\mathcal{L}$ and and its commutator, which can be seen in [4].

Definition 1.1. Let $\mathcal{L}=-\Delta+V$ with $V\in RH_{n/2}$ for $q\ge n/2$, the $\mathcal{L}$-fractional integral operator is defined by

$$\mathcal{I}_\beta f(x)=\mathcal{L}^{-\frac{\beta}{2}}f(x)=\int_0^\infty e^{-t\mathcal{L}}f(x)t^{-\frac{\beta}{2}-1}dt$$

for $0<\beta<n$. Let $b\in BMO$, the commutator of $\mathcal{I}_\beta$ is defined by

$$[b,\mathcal{I}_\beta]f(x)=b\mathcal{I}_\beta f(x)-\mathcal{I}_\beta(bf)(x).$$

We can formulate our results as follows.

Theorem 1.1. *Suppose $V\in RH_{n/2}$, $\alpha\in(-\infty,\infty)$, and $0<\beta<n$. If*

$$1<p<\frac{n}{\beta},\quad \frac{1}{q}=\frac{1}{p}-\frac{n}{\beta},\quad \frac{1}{q_1}=\frac{1}{p_1}-\frac{n}{\beta},$$

then

$$\|\mathcal{I}_\beta f\|_{M_{\alpha,v}^{q,q_1}}\le C\|f\|_{M_{\alpha,v}^{p,p_1}}.$$

Theorem 1.2. *Let $b\in BMO$, $V\in RH_{n/2}$, $\alpha\in(-\infty,\infty)$, and $0<\beta<n$. If*

$$1<p<\frac{n}{\beta},\quad \frac{1}{q}=\frac{1}{p}-\frac{n}{\beta},\quad \frac{1}{q_1}=\frac{1}{p_1}-\frac{n}{\beta},$$

then

$$\|[b,\mathcal{I}_\beta f]\|_{M_{\alpha,v}^{q,q_1}}\le C\|f\|_{M_{\alpha,v}^{p,p_1}}.$$