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Complex and p-Adic Meromorphic Functions $f'P'(f)$, $g'P'(g)$ Sharing a Small Function

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Abstract. Let $\mathbb{K}$ be a complete algebraically closed p-adic field of characteristic zero. We apply results in algebraic geometry and a new Nevanlinna theorem for p-adic meromorphic functions in order to prove results of uniqueness in value sharing problems, both on $\mathbb{K}$ and on $\mathbb{C}$. Let P be a polynomial of uniqueness for meromorphic functions in $\mathbb{K}$ or $\mathbb{C}$ or in an open disk. Let f, g be two transcendental meromorphic functions in the whole field $\mathbb{K}$ or in $\mathbb{C}$ or meromorphic functions in an open disk of $\mathbb{K}$ that are not quotients of bounded analytic functions. We show that if $f'P'(f)$ and $g'P'(g)$ share a small function α counting multiplicity, then $f=g$, provided that the multiplicity order of zeros of P' satisfy certain inequalities. A breakthrough in this paper consists of replacing inequalities $n \geq k+2$ or $n \geq k+3$ used in previous papers by Hypothesis (G). In the p-adic context, another consists of giving a lower bound for a sum of q counting functions of zeros with $(q-1)$ times the characteristic function of the considered meromorphic function.

Key Words: Meromorphic, nevanlinna, sharing value, unicity, distribution of values.

AMS Subject Classifications: 12J25, 30D35, 30G06

1 Introduction

Notation and Definition 1.1. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete with respect to an ultrametric absolute value $|\cdot|$. We will denote by $\mathbb{E}$ a field that is either $\mathbb{K}$ or $\mathbb{C}$. Throughout the paper we denote by a a point in $\mathbb{K}$. Given $R \in [0,+\infty]$ we define disks $d(a,R)=\{x \in \mathbb{K} \mid |x-a| \leq R\}$ and disks $d(a,R^-)=\{x \in \mathbb{K} \mid |x-a| < R\}$.

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A polynomial $Q(X) \in \mathbb{E}[X]$ is called *a polynomial of uniqueness for a family of functions $\mathcal{F}$ defined in a subset of* $\mathbb{E}$ if $Q(f) = Q(g)$ implies $f = g$. The definition of polynomials of uniqueness was introduced in [19] by P. Li and C. C. Yang and was studied in many papers [11,13,20] for complex functions and in [1,2,9,10,17,18], for p-adic functions.

Throughout the paper we will denote by $P(X)$ a polynomial in $\mathbb{E}[X]$ such that $P'(X)$ is of the form $b\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $l \geq 2$ and $k_1 \geq 2$. The polynomial P will be said *to satisfy Hypothesis (G)* if $P(a_i) + P(a_j) \neq 0$, $\forall i \neq j$.

We will improve the main theorems obtained in [5] and [6] with the help of a new hypothesis denoted by Hypothesis (G) and by thorougly examining the situation with p-adic and complex analytic and meromorphic functions in order to avoid a lot of exclusions. Moreover, we will prove a new theorem completing the 2nd Main Theorem for p-adic meromorphic functions. Thanks to this new theorem we will give more precisions in results on value-sharing problems.

Notation 1.1. Let L be an algebraically closed field, let $P \in L[x] \setminus L$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c. We denote by $\Phi(P)$ its cardinal.

We denote by $\mathcal{A}(\mathbb{E})$ the $\mathbb{E}$-algebra of entire functions in $\mathbb{E}$, by $\mathcal{M}(\mathbb{E})$ the field of meromorphic functions in $\mathbb{E}$, i.e., the field of fractions of $\mathcal{A}(\mathbb{E})$ and by $\mathbb{E}(x)$ the field of rational functions. Throughout the paper, we denote by $\mathcal{A}(d(a,R^-))$ the $\mathbb{K}$-algebra of analytic functions in $d(a,R^-)$ i.e., the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converging in $d(a,R^-)$ and we denote by $\mathcal{M}(d(a,R^-))$ the field of meromorphic functions inside $d(a,R^-)$, i.e., the field of fractions of $\mathcal{A}(d(a,R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a,R^-))$ the $\mathbb{K}$-subalgebra of $\mathcal{A}(d(a,R^-))$ consisting of the bounded analytic functions in $d(a,R^-)$, i.e., which satisfy $\sup_{n\in\mathbb{N}}|a_n|R^n < +\infty$. We denote by $\mathcal{M}_b(d(a,R^-))$ the field of fractions of $\mathcal{A}_b(d(a,R^-))$ and finally, we denote by $\mathcal{A}_u(d(a,R^-))$ the set of unbounded analytic functions in $d(a,R^-)$, i.e., $\mathcal{A}(d(a,R^-)) \setminus \mathcal{A}_b(d(a,R^-))$. Similarly, we set $\mathcal{M}_u(d(a,R^-)) = \mathcal{M}(d(a,R^-)) \setminus \mathcal{M}_b(d(a,R^-))$.

Theorem 1.1 (see [9]). *Let $P(X) \in \mathbb{K}[X]$. If $\Phi(P) \geq 2$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ and for $\mathcal{A}_u(d(a,R^-))$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}_u(d(a,R^-))$.*

Let $P(X) \in \mathbb{C}[X]$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.

Concerning polynomials such that P' has exactly two distinct zeros, we know other results:

Theorem 1.2 (see [1,2,18]). *Let $P \in \mathbb{K}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 with* $\min\{c_1,c_2\} \geq 2$. *Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.*

Theorem 1.3 (see [9,17]). *Let $P \in \mathbb{K}[x]$ be of degree $n \geq 6$ be such that P' only has two distinct zeros, one of them being of order* 2. *Then P is a polynomial of uniqueness for $\mathcal{M}_u(d(0,R^-))$.*