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On the Negative Extremums of Fundamental Functions of Lagrange Interpolation Based on Chebyshev Nodes

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Abstract. In this paper, we investigate the negative extremums of fundamental functions of Lagrange interpolation based on Chebyshev nodes. Moreover, we establish some companion results to the theorem of J. Szabados on the positive extremum.

Key Words: Negative extremum, Lagrange interpolation, Chebyshev polynomial, fundamental function of interpolation.

AMS Subject Classifications: 41A05, 41A10

1 Introduction

Let $T_n(x)=\cos(n\arccos x)$ be the Chebyshev polynomial of degree n, with roots

$$x_{k,n}=\cos t_{k,n},\quad t_{k,n}=\frac{2k-1}{2n}\pi,\quad k=1,2,\cdots,n.$$

And let

$$l_{k,n}(x)=\frac{(-1)^{k-1}\cos nt\sin t_{k,n}}{n(\cos t-\cos t_{k,n})},\quad x=\cos t,\quad k=1,2,\cdots,n, \tag{1.1}$$

be the fundamental polynomials of Lagrange interpolation based on Chebyshev nodes. Furthermore, we define

$$\begin{aligned}
&\overline{l}_{k,n}=\max_{|x|\le 1} l_{k,n}(x), && \underline{l}_{k,n}=\min_{|x|\le 1} l_{k,n}(x), && k=1,2,\cdots,n,\\
&M_n(x)=\max_{1\le k\le n} l_{k,n}(x), && m_n(x)=\min_{1\le k\le n} l_{k,n}(x), && |x|\le 1,
\end{aligned}$$

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$$\overline{M}_n = \max_{|x|\le 1} M_n(x), \qquad \underline{M}_n = \min_{|x|\le 1} M_n(x),$$

$$\overline{M}_n^* = \max_{1\le k\le n} \bar{l}_{k,n}, \qquad \underline{M}_n^* = \min_{1\le k\le n} \bar{l}_{k,n},$$

$$\overline{m}_n = \max_{|x|\le 1} m_n(x), \qquad \underline{m}_n = \min_{|x|\le 1} m_n(x),$$

$$\overline{m}_n^* = \max_{1\le k\le n} \underline{l}_{k,n}, \qquad \underline{m}_n^* = \min_{1\le k\le n} \underline{l}_{k,n}.$$

In [1], Erdös and Grünwald proved the following theorem.

Theorem 1.1. *We have*

$$|l_{k,n}(x)| < \frac{4}{\pi}, \quad |x|\le 1, \quad 1\le k\le n, \quad n=1,2,\cdots. \tag{1.2}$$

Moreover,

$$\lim_{n\to\infty} l_{1,n}(1) = \lim_{n\to\infty} l_{n,n}(-1) = \frac{4}{\pi}. \tag{1.3}$$

It follows from Theorem 1.1 that

$$\lim_{n\to\infty} \overline{M}_n = \lim_{n\to\infty} \overline{M}_n^* = \frac{4}{\pi}.$$

In [2], J. Szabados proved the following theorem.

Theorem 1.2. *We have*

$$\lim_{n\to\infty} \underline{M}_n = \frac{2}{\pi}\cos\frac{2-\sqrt{3}}{2}\pi = 0.580\cdots. \tag{1.4}$$

In [3], Laiyi Zhu and Yang Tan proved the following theorem.

Theorem 1.3. *We have*

$$\lim_{n\to\infty} \underline{M}_n^* = 1. \tag{1.5}$$

It is natural to ask for the behavior of $\overline{m}_n$, $\underline{m}_n$, $\overline{m}_n^*$ and $\underline{m}_n^*$, which is of some interest to J. Szabados (see [2] for details). In this paper, we investigate the negative extremums of $l_{k,n}(x)$ and obtain the limits of $\overline{m}_n$, $\underline{m}_n$, $\overline{m}_n^*$ and $\underline{m}_n^*$.

2 The negative extremums of $l_{k,n}(x)$

In this section, we shall investigate the negative extremums of $l_{k,n}(x)$, $k=1,2,\cdots,n$. For the sake of convenience, as n is fixed, we denote $t_{k,n}$, $x_{k,n}$, $l_{k,n}(x)$, $\bar{l}_{k,n}$ and $\underline{l}_{k,n}$ by t_k, x_k, $l_k(x)$, $\bar{l}_k$ and $\underline{l}_k$ respectively, and denote $l_k(\cos t)$ by $I_k(t)$, $k=1,2,\cdots,n$. By symmetry, it is sufficient to consider the cases $1\le k\le [(n+1)/2]$, where $[t]$ denotes the integer part of t.