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Boundedness for Multilinear Commutators of Marcinkiewicz Integral on Morrey-Herz Spaces with Non Doubling Measures

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Abstract. In this paper, the authors establish the boundedness of multilinear commutators generated by a Marcinkiewicz integral operator and a RBMO(μ) function on homogeneous Morrey-Herz spaces with non doubling measures.

Key Words: Marcinkiewicz integral, commutator, Morrey-Herz space, non doubling measure, RBMO function.

AMS Subject Classifications: 47B47, 42B20, 47A30

1 Introduction and preliminaries

As an analogy of the classical Littlewood-Paley g function, Marcinkiewicz [1] introduced the operator

$$\mathcal{M}(f)(x)=\Big(\int_0^{\pi}\frac{|F(x+t)+F(x-t)-2F(x)|^2}{t^3}\mathrm{d}t\Big)^{\frac{1}{2}},\quad x\in[0,2\pi],$$

where $F(x)=\int_0^x f(t)\mathrm{d}t$. This operator is now called the Marcinkiewicz integral. Zygmund [2] proved that the operator $\mathcal{M}$ is bounded on the Lebesgue space $L^p([0,2\pi])$ for $p\in(1,\infty)$. Stein [3] generalized the above Marcinkiewicz integral to the following higher-dimensional case. Let Ω be homogeneous of degree zero in $\mathbf{R}^d$ for $d\geq 2$, integrable and have mean value zero on the unit sphere S^{d-1}. The higher-dimensional Marcinkiewicz integral is defined by

$$\mathcal{M}_\Omega(f)(x)=\Big(\int_0^{\infty}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{d-1}}f(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}},\quad x\in\mathbf{R}^d.$$

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Stein in [3] proved that if $\Omega\in\mathrm{Lip}_\delta(S^{d-1})$ for some $\delta\in(0,1]$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^d)$ for any $p\in(1,2]$, and is also bounded from $L^1(\mathbf{R}^d)$ to $L^{1,\infty}(\mathbf{R}^d)$. Since then, a lot of papers focus on this operator. For some recent development, we mention that Al-Salman et al. in [4] obtained the $L^p(\mathbf{R}^d)$-boundedness for $p\in(1,\infty)$ of $\mathcal{M}_\Omega$ if $\Omega\in L(\log L)^{1/2}(S^{d-1})$; Fan and Sato in [5] proved that $\mathcal{M}_\Omega$ is bounded from the Lebesgue space $L^1(\mathbf{R}^d)$ to the weak Lebesgue space $L^{1,\infty}(\mathbf{R}^d)$ if $\Omega\in L\log L(S^{d-1})$. There are many other interesting works for this operator, among them we refer to [6, 7] and their references. On the other hand, Torchinsky and Wang in [8] first introduced the commutator generated by the Marcinkiewicz integral $\mathcal{M}_\Omega$ and the classical $\mathrm{BMO}(\mathbf{R}^d)$ function, and established its $L^p(\mathbf{R}^d)$-boundedness for $p\in(1,\infty)$ when $\Omega\in\mathrm{Lip}_\delta(S^{d-1})$ for some $\delta\in(0,1]$. Such boundedness of this commutator is further discussed in [9, 10] when Ω only satisfies certain size conditions. Moreover, its weak type endpoint estimate is obtained in [11, 12] when $\Omega\in\mathrm{Lip}_\delta(S^{d-1})$ for some $\delta\in(0,1]$, and its weight weak type endpoint estimate is obtained in [13, 14] when Ω satisfies a kind of Dini conditions. Also see [15–17] et al. for more informations.

Motivated by the work above, the main purpose of this paper is to establish a similar theory for the multilinear commutator generated by a Marcinkiewicz integral operator and a $\mathrm{RBMO}(\mu)$ function or $\mathrm{Osc}_{\exp L^r}(\mu)$ function on $\mathbf{R}^d$ with a positive Radon measure which may be non doubling.

To be precise, let μ be a positive Radon measure on $\mathbf{R}^d$ which only satisfies the following growth condition that for all $x\in\mathbf{R}^d$ and all $r>0$,

$$\mu(B(x,r))\le C_0r^n, \tag{1.1}$$

where $C_0>0$ and n are some positive constants, $0<n\le d$, and $B(x,r)$ is the open ball centered at x and having radius r. We recall that μ is said to be a doubling measure, if there is a positive constant C such that for any $x\in\operatorname{supp}\mu$ and $r>0$,

$$\mu(B(x,2r))\le C\mu(B(x,r)),$$

and that the doubling condition is a key assumption in the classical theory of harmonic analysis. In recent years, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved to be still valid if the Lebesgue measure is substituted by a measure μ as in (1.1); see [18–25]. We mention that the analysis on non-homogeneous spaces play an essential role in solving the long-standing open Painlevé's problem by Tolsa in [21].

To outline the structure of this paper, we first recall some notation and definitions. For a cube $Q\subset\mathbf{R}^d$, we mean a closed cube whose sides parallel to the coordinate axes, and we denote its side length by $l(Q)$ and its center by x_Q. Let $\gamma>1$ and $\beta>\gamma^n$. We say that a cube Q is an (γ,β)-doubling cube if $\mu(\gamma Q)\le\beta\mu(Q)$, where γQ denotes the cube with the same center as Q and $l(\gamma Q)=\gamma l(Q)$. For definiteness, if γ and β are not specified, by a doubling cube we mean a $(2,2^{d+1})$-doubling cube. Especially, for any given cube Q, we denote by $\tilde{Q}$ the smallest doubling cube which contains Q and has the same center as Q.