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Some Results Concerning Growth of Polynomials

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Abstract. Let $P(z)$ be a polynomial of degree n having no zeros in $|z|<1$, then for every real or complex number β with $|\beta|\leq 1$, and $|z|=1$, $R\geq 1$, it is proved by Dewan et al. [4] that

$$\Big|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^nP(z)\Big|\leq\frac{1}{2}\Big\{\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big)\max_{|z|=1}|P(z)| \\ -\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|-\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n|\Big)\min_{|z|=1}|P(z)|\Big\}.$$

In this paper we generalize the above inequality for polynomials having no zeros in $|z|<k$, $k\leq 1$. Our results generalize certain well-known polynomial inequalities.

Key Words: Polynomial, inequality, maximum modulus, growth of polynomial.

AMS Subject Classifications: 30A10, 30C10, 30E15

1 Introduction and statement of results

It is well known that if $P(z)$ is a polynomial of degree n, then for $|z|=1$ and $R\geq 1$

$$|P(Rz)|+|Q(Rz)|\leq (R^n+1)\max_{|z|=1}|P(z)|, \tag{1.1}$$

where $Q(z)=z^n\overline{P(1/\overline{z})}$ (see [6]).

On the other hand, concerning the estimate of $|P(z)|$ on the disc $|z|\leq R$, $R\geq 1$, we have, as a simple consequence of the principle of maximum modulus (see also [6]), if $P(z)$ is a polynomial of degree n, then for $R\geq 1$

$$\max_{|z|=R}|P(z)|\leq R^n\max_{|z|=1}|P(z)|. \tag{1.2}$$

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The result is best possible and the equality holds for polynomials having zeros at the origin.

It was shown by Ankeny and Rivlin [1] that if $P(z)$ doe not vanish in $|z|<1$, then the inequality (1.2) can be replaced by

$$\max_{|z|=R}|P(z)|\leq\frac{R^n+1}{2}\max_{|z|=1}|P(z)|,\quad R\geq 1. \tag{1.3}$$

The inequality (1.3) is sharp and the equality holds for $P(z)=\alpha z^n+\gamma$, where $|\alpha|=|\gamma|$.

The inequality (1.3) was generalized by Jain [5] who proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then for $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$,

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^nP(z)\Big|\\&\leq\frac{1}{2}\Big\{\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big\}\max_{|z|=1}|P(z)|.\end{aligned} \tag{1.4}$$

Aziz and Dawood [3] used

$$\min_{|z|=1}|P(z)| \tag{1.5}$$

to obtain a refinement of the inequality (1.3) and proved, if $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for $R\geq 1$

$$\max_{|z|=R}|P(z)|\leq\Big(\frac{R^n+1}{2}\Big)\max_{|z|=1}|P(z)|-\Big(\frac{R^n-1}{2}\Big)\min_{|z|=1}|P(z)|. \tag{1.6}$$

The result is best possible and the equality holds for $P(z)=\alpha z^n+\gamma$ with $|\alpha|=|\gamma|$.

As refinement of the inequality (1.4) and generalization of the inequality (1.6), Dewan and Hans [4] have proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then for $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$,

$$\begin{aligned}\Big|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^nP(z)\Big|\leq\frac{1}{2}\Big\{&\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big)\max_{|z|=1}|P(z)|\\&-\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|-\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big)\min_{|z|=1}|P(z)|\Big\}.\end{aligned} \tag{1.7}$$

The result is best possible and the equality holds for $P(z)=\alpha z^n+\gamma$ with $|\alpha|=|\gamma|$.

Whereas if $P(z)$ has all its zeros in $|z|\leq 1$, then for any $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$,

$$\min_{|z|=1}\Big|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^nP(z)\Big|\geq\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\min_{|z|=1}|P(z)|. \tag{1.8}$$

The result is best possible and the equality holds for $P(z)=me^{i\alpha}z^n$, $m>0$.

In this paper, we obtain further generalizations of the inequalities (1.7) and (1.8). More precisely, we prove