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THE BOUNDEDNESS OF TOEPLITZ-TYPE OPERATORS ON VANISHING-MORREY SPACES

Xiaoniu Cao and Dongxiang Chen

(*Jiangxi Normal University, China)*

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Abstract. In this note, we prove that the Toeplitz-type Operator Θ_α^b generated by the generalized fractional integral, Calderón-Zygmund operator and VMO funtion is bounded from $L^{p,\lambda}(R^n)$ to $L^{q,\mu}(R^n)$. We also show that under some conditions $\Theta_\alpha^b f \in VL^{q,\mu}(B_R)$, the vanishing-Morrey space.

Key words: *Toeplitz-type operator, generalized fractional integral, vanishing-Morrey space*

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1 Introduction and Main Result

Suppose that L is a linear operator on $L^2(\mathbf{R}^n)$, which generates an analytic semigroup e^{-tL} with a kernel $p_t(x,y)$ satisfying a Gaussian kernel bound, that is,

$$|p_t(x,y)| \le \frac{C}{t^{\frac{n}{2}}} e^{-c\frac{|x-y|^2}{t}}, \tag{1.1}$$

for $x,y \in \mathbf{R}^n$ and all $t>0$.

For $0<\alpha<n$, the generalized fractional integral $L^{-\alpha/2}$ generated by the operator L is defined by

$$L^{-\alpha/2}f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f) \frac{\mathrm{d}t}{t^{-\alpha/2+1}}(x). \tag{1.2}$$

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When $L = \triangle$ is the Laplacian operator on $\mathbf{R}^n$, $L^{-\alpha/2}$ is the classical fractional integral I_α, for example see [1], which is given by

$$I_\alpha f(x) = \frac{\Gamma((n-\alpha)/2)}{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}\int_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y.$$

In 1982, S. Chanillo[2] showed that for all $0<\alpha<n$ and $b\in \mathrm{BMO}(\mathbf{R}^n)$, the commutator $[b,I_\alpha]$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ with $1<p<n/\alpha, 1/q = 1/p-\alpha/n$. In 2004, Duong and Yan [3] proved that for all $0<\alpha<n$ and $b\in \mathrm{BMO}$, both $L^{-\alpha/2}$ and the commutator $[b,L^{-\alpha/2}]$ are bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, where $1<p<n/\alpha$, $1/q = 1/p-\alpha/n$. If $b\in \mathrm{BMO}(\mathbf{R}^n)$, the commutator $T^b f = bTf - T(bf)$, T is a Calderón-Zygmund operator with a standard kernel K , we know that T^b is (L^p,L^p)-boundedness for $1<p<\infty$.

In fact, since the kernel of $L^{-\alpha/2}$ is $K_\alpha(x,y)$ and the kernel of e^{-tL} is $p_t(x,y)$, which satisfies (1.1), we have

$$L^{-\alpha/2}f(x) = \int_{R^n}K_\alpha(x,y)f(y)\mathrm{d}y,$$

thus

$$K_\alpha(x,y) = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty p_t(x,y)\frac{\mathrm{d}t}{t^{-\alpha/2+1}}. \tag{1.3}$$

And using(1.1),

$$|K_\alpha(x,y)| \le C\frac{\Gamma(n/2-\alpha/2)}{\Gamma(\alpha/2)}\frac{1}{|x-y|^{n-\alpha}}, \tag{1.4}$$

for $x\neq y$ and if $|x-z|\ge 2|y-z|$,

$$|K_\alpha(x,y)-K_\alpha(x,z)| + |K_\alpha(y,x)-K_\alpha(z,x)| \le C\frac{\Gamma(n/2-\alpha/2)}{\Gamma(\alpha/2)}\frac{|y-z|}{|x-z|}|x-z|^{\alpha-n}. \tag{1.5}$$

Let $B = B(x,\rho)$ be a ball in R^n of radius ρ at the point x.

Definition 1.1. Given $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, let us set

$$Mf(x) = \sup_{x\in B}\frac{1}{|B|}\int_B|f(y)|\mathrm{d}y, \qquad for \qquad \text{a. e. } x\in\mathbf{R}^n.$$

M is the Hardy-Littlewood maximal operator.

Define the Sharp maximal function by

$$f^\sharp(x) = \sup_{x\in B}\frac{1}{|B|}\int_B|f(y)-f_B|\mathrm{d}y, \qquad for \qquad \text{a. e.} \quad x\in\mathbf{R}^n.$$

Definition 1.2. Let $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $0<\eta<1$, we set

$$M_\eta f(x) = \sup_{x\in B}\frac{1}{|B|^{1-\eta}}\int_B|f(y)|\mathrm{d}y, \qquad for \qquad \text{a. e.} \quad x\in\mathbf{R}^n.$$