Anal. Theory Appl.
Vol. 27, No. 3 (2011), 220–223
DOI10.1007/s10496-011-0220-6

SOME APPLICATIONS OF BP-THEOREM IN APPROXIMATION THEORY

I. Sadeqi

(*Sahand University of Technology, Iran*)

R. Zarghami

(*University of Tabriz, Iran*)

Received May 31, 2010

Abstract. In this paper we apply Bishop-Phelps property to show that if X is a Banach space and $G \subseteq X$ is the maximal subspace so that $G^{\perp} = \{x^* \in X^* | x^*(y) = 0; \quad \forall y \in G\}$ is an L-summand in X^*, then $L^1(\Omega, G)$ is contained in a maximal proximinal subspace of $L^1(\Omega, X)$.

Key words: *Bishop-Phelps theorem, support point, proximinality, L-projection*

AMS (2010) subject classification: 46E99

1 Introduction

In the note we need some definitions and notations which are the following. Let (Ω, Σ, μ) be a measure space with non-negative complete σ-finite measure μ and σ-algebra Σ of μ-measurable sets. We denote by $L_P(\Omega, \Sigma, \mu : X) = L_p(\Omega, X)$ the Banach space of all equivalence classes of all Bochner integrable functions $f : \Omega \to X$ with the norm

$$\|f\| = (\int_{\Omega} \|f(t)\|^p \mathrm{d}\mu)^{\frac{1}{p}}; 1 \le p < \infty,$$

$$\|f\|_{\infty} = \operatorname*{ess\,sup}_{t \in \Omega} \|f(t)\|; p = \infty.$$

A set M of measurable functions $f : \Omega \to X$ is decomposable if for any two elements f , g in M and $E \in \Sigma$, $\chi_E f + \chi_{\Omega \setminus E} g \in M$, where χ_E is the characteristic function of E. Let X be a real or complex Banach space and C be a closed convex subset of X. The set of support points of C is the collection of all points $z \in C$ for which there exists nontrivial $f \in X^*$ such that $\sup_{x\in C}|f(x)| = |f(z)|$. Such an f is called support functional. The support point z is said to be exposed, if Re $f(x) <$ Re$f(z)$, for $x(\neq z) \in C$. We denote by Supp C and ΣC the set of support points and support functionals, respectively. Bishop and Phelps[5] have shown that if C is a closed convex and bounded subset of X then Supp C is dense in the boundary of C and ΣC is dense in X^*. The complex case of the Bishop-Phelps theorem is also studied in [6] and some results are given.

Let X be a Banach space and G a closed subspace of X. The subspace G is called proximinal in X if for every $x \in X$ there exists at least one $y \in G$ such that

$$||x-y|| = \inf\{||x-z|| : z \in G\}.$$

A linear projection P is called an L-projecton if

$$\|x\| = \|Px\| + \|x-Px\|; \qquad \forall x \in X.$$

A closed subspace $Y \subset X$ is called an $L-summand$ if it is the range of an $L-projection$.

The natural question is that, whether or not $L^1(\Omega, G)$ is proximinal in $L^1(\Omega, X)$ if G is proximinal in X [3].We will show that if $G^\perp$ is an $L-summand$ then $L^1(\Omega, G)$ is contained in a maximal proximinal subspace of $L^1(\Omega, X)$.

2 The Main Results

Theorem 2.1[4]**.** *If X is a Banach space and $T \in X^*$, then kerT is a proximinal set in X if and only if T supports some points of the unit ball of X.*

Lemma 2.2. *Let X be a Banach space and G a support set in X. Suppose $L^1(\Omega, G)$ is a decomposable set. Then each constant function of $L^1(\Omega, G)$ is a support point for $L^1(\Omega, G)$.*

Proof. Let $g_0 \in L^1(\Omega, G)$ be a constant function, then there exists a point $x_0 \in G$ such that $g_0(t) = x_0$. Since G is a support set, we have

$$\exists T_0 \in X^* \text{ s. t. } \inf_G T_0 = T_0(x_0).$$

We define $F_0 : L^1(\Omega, X) \to R$ as follows:

$$F_0(g) = \int_\Omega T_0(g(t))\mathrm{d}\mu.$$

It is obvious that $F_0 \in L^1(\Omega, X)^*$, because if

$$g_n \to g \quad (||g_n - g|| \to 0),$$