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A NEW BLO ESTIMATE FOR MAXIMAL SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, we extend Hu and Zhang's results in [2] to different case.

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1 Introduction

We will work on $\mathbf{R}^n$, $n \geq 2$. Let Ω be homogeneous of degree zero, integrable on the unit sphere S^{n-1} and have mean value zero. Define the singular integral operator T by

$$Tf(x) = p.v. \int_{R^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) \mathrm{d}y \tag{1.1}$$

and the corresponding maximal operator T^* by

$$T^* f(x) = \sup_{0<\varepsilon<N<\infty} |T_{\varepsilon,N} f(x)|, \tag{1.2}$$

where $T_{\varepsilon,N} f(x)$ is the truncated operator defined by

$$T_{\varepsilon,N} f(x) = \int_{\varepsilon<|x-y|\leq N} \frac{\Omega(x-y)}{|x-y|^n} f(y) \mathrm{d}y. \tag{1.3}$$

Definition 1. The space $\mathrm{BLO}(R^n)$ consists of all $f \in L^1_{\mathrm{Loc}}(\mathbf{R}^n)$ such that

$$\|f\|_{\mathrm{BLO}(R^n)} = \sup_B (m_B(f) - \inf_{x\in B} f(x)) < \infty,$$

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where the supremum is taken over all balls B and $m_B(f)$ denotes the mean value of f on the ball B, that is, $m_B(f) = \frac{1}{|B|}\int_B f(x)\mathrm{d}x$.

Definition 2. Let $\Omega \in L^1(S^{n-1})$, define the L^1 modulus of continuity of Ω as

$$\omega(\delta) = \sup_{|\rho|\le\delta}\int_{S^{n-1}} |\Omega(\rho x) - \Omega(x)|\mathrm{d}\sigma(x),$$

where $|\rho|$ denotes the distance of ρ from the identity rotation, and the supremum is taken over all rotations on the unit sphere with $|\rho| \le \delta$.

Definition 3. As usual, a function $A : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, conex and increasing satisfying $A(0) = 0$ and $A(t) \to \infty$ as $t \to \infty$. We define the A-average of a function f over a ball B by means of the following Luxemburg norm

$$\|f\|_{A,B} = \inf\{\lambda > 0 : \frac{1}{|B|}\int_B A\left(\frac{|f(y)|}{\lambda}\right)\mathrm{d}y \le 1\}. \tag{1.4}$$

The following generalized Hölder's inequality holds:

$$\frac{1}{|B|}\int_B |f(y)g(y)|\mathrm{d}y \le \|f\|_{A,B}\|g\|_{A_1,B}, \tag{1.5}$$

where A_1 is the complementary function associated to A (see[4][5]).

Definition 4. For a suitable Young function A and its complementary function A_1, we say f satisfies A_1^q-condition if it satisfies

$$\frac{1}{|B|}\int_B A_1\left(\frac{|f(y) - m_B(f)|^q}{C}\right)\mathrm{d}y \le C_1,$$

where $q \ge 1$, C and C_1 are positive constants.

For a Young function $A(t) = t\log(2+t)$, its complementary function $A_1(t) \approx \exp t$, Hu Guoen and Zhang Qihui[2] proved the following theorem:

Theorem A. Let T^* be the maximal singular integrable operator defined by (1.2), Ω be homogeneous of degree zero, integrable on the unit sphere S^{n-1} and have mean value zero. Suppose that for some $q > 2$, $\Omega \in L(\log L)^q(S^{n-1})$, namely,

$$\int_{S^{n-1}} |\Omega|\log^q(2+|\Omega|)\mathrm{d}\sigma(x) < \infty,$$

and the L^1 modulus of continuity of Ω satisfies

$$\int_0^1 \omega(\delta)\log(2+\frac{1}{\delta})\frac{\mathrm{d}\delta}{\delta} < \infty.$$

Then for any $f \in$BMO($\mathbf{R}^n$), $T^*f(x)$ is either infinite everywhere or finite almost everywhere. More precise, if $f \in$BMO($\mathbf{R}^n$) such that $T^*f(x_0) < \infty$ for some $x_0 \in \mathbf{R}^n$, then $T^*f(x)$ is finite almost everywhere, and

$$\|T^*f\|_{\mathrm{BLO}(R^n)} \le C\|f\|_{\mathrm{BMO}(R^n)}.$$