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A HYBRID FIXED POINT RESULT FOR LIPSCHITZ HOMOMORPHISMS ON QUASI-BANACH ALGEBRAS

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Abstract. We shall generalize the results of [9] about characterization of isomorphisms on quasi-Banach algebras by providing integral type conditions. Also, we shall give some new results in this way and finally, give a result about hybrid fixed point of two homomorphisms on quasi-Banach algebras.

Key words: *homomorphism, hybrid fixed point, integral-type condition, p-norm, quasi-Banach algebra*

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1 Introduction

The stability problem of functional equations originated from a question of Ulam[12] concerning the stability of group homomorphisms: Let $(G_1,*)$ be a group and $(G_2,\diamond,d)$ be a metric group. Given $\varepsilon>0$, does there exist $\delta(\varepsilon)>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$? If the answer is affirmative, we would say that the equation of the homomorphism $H(x*y)=H(x)\diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers[7] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f:X\to Y$ satisfies

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon$$

for all $x, y \in X$ and some $\varepsilon \geq 0$. Then, there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in X$.

Let X and Y be Banach spaces and $f : X \to Y$ a mapping such that $f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in X$. Th. M. Rassias[10] introduced the following inequality: Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. He proved that there exists a unique $\mathbf{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in X$. The above inequality has provided a lot of influence in the development of what is now known as Hyers-Ulam-Rassias stability of functional equations and there are a lot of works in this field. In 2007, Park, Cho and Han[8] proved the Hyers-Ulam-Rassias stability of functional inequalities associated with Jordan-Von Neumann type additive functional equations. Then, Park and An characterized isomorphisms in quasi-Banach algebras in this way.

On the other hand, Hybrid fixed point theory is an important topic and there are some papers in this field (see for example [3]-[6]). In this paper, we shall generalize the results of [9] about characterization of isomorphisms on quasi-Banach algebras by providing integral type conditions. Also, we shall give some new results in this way and finally give a result about hybrid fixed point of two homomorphisms on quasi-Banach algebras. Here, we recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1[2],[11]. Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following conditions:

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.

(2) $||\lambda x|| = |\lambda| \cdot ||x||$ for all $\lambda \in \mathbf{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $||x + y|| \leq K(||x|| + ||y||)$ for all $x, y \in X$.

The pair $(X, || \cdot ||)$ is called a *quasi-normed* space if $|| \cdot ||$ is a quasi-normed on X.

A quasi-Banach space is a complete quasi-normed space.

Definition 1.2[1]. Let $(A, || \cdot ||)$ be a quasi-normed space. The quasi-normed space $(A, || \cdot ||)$ is called a *quasi-normed algebra* if A is an algebra and there is a constant $K > 0$ such that $||xy|| \leq K||x|| \cdot ||y||$ for all $x, y \in A$.

A quasi-Banach algebra is a complete quasi-normed algebra.

Definition 1.3[9]. A $\mathbf{C}$-linear mapping $H : A \to B$ is called a homomorphism on quasi-normed algebras if $H(xy) = H(x)H(y)$ for all $x, y \in A$. If in addition, the mapping $H : A \to B$ is bijective, then the mapping $H : A \to B$ is called an isomorphism on quasi-normed algebras.