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Numerical Inversion for the Initial Distribution in the Multi-Term Time-Fractional Diffusion Equation Using Final Observations

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Abstract. This article deals with numerical inversion for the initial distribution in the multi-term time-fractional diffusion equation using final observations. The inversion problem is of instability, but it is uniquely solvable based on the solution's expression for the forward problem and estimation to the multivariate Mittag-Leffler function. From view point of optimality, solving the inversion problem is transformed to minimizing a cost functional, and existence of a minimum is proved by the weakly lower semi-continuity of the functional. Furthermore, the homotopy regularization algorithm is introduced based on the minimization problem to perform numerical inversions, and the inversion solutions with noisy data give good approximations to the exact initial distribution demonstrating the efficiency of the inversion algorithm.

AMS subject classifications: 35R11, 35R30, 65M06

Key words: Multi-term time-fractional diffusion, multivariate Mittag-Leffler function, backward problem, ill-posedness, numerical inversion.

1 Introduction

The fractional diffusion equations have played an important role in modeling of the anomalous diffusion phenomena and in the theory of complex systems (see e.g., [1, 3–5, 9, 31], and references therein) instead of the classical diffusion equations during the last few decades. The so-called time fractional diffusion equation arises when replacing the first-order time derivative by a fractional derivative of order α with $0<\alpha<1$. On the other hand, by the attempts to describe some real processes with the fractional diffusion equations, some researches confronted with the situation that the time-fractional derivative from the corresponding models did not remain constant and changed, say, in the

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interval from 0 to 1, from 1 to 2 or even from 0 to 2. To manage these phenomena, several approaches were suggested. One of them introduces the fractional derivatives of the variable order, i.e., the derivatives with the order that can change with the time or/and depending on the spatial coordinates, and the other way is to employ the time-fractional diffusion equations of distributed order, in which the so-called multi-term time-fractional diffusion equation is an important particular case (see e.g., [8,11,12,18,22–24,28,29], and references therein). A general multi-term time fractional diffusion equation is given as

$$P({}_0^C D_t)u(x,t)=L_x(u(x,t))+F(x,t), \quad (x,t)\in\Omega_T:=\Omega\times(0,T], \tag{1.1}$$

where $\Omega\subset\mathbf{R}^d$ ($d\ge 1$) is an open bounded domain with smooth boundary, $F(x,t)$ is a source term, and L_x is a symmetric uniformly elliptic operator given by

$$L_x(u):=\mathrm{div}(p(x)\nabla u)-q(x)u, \quad x\in\Omega, \tag{1.2}$$

in which the coefficients satisfy

$$p\in C^1(\overline{\Omega}), \quad q\in C(\overline{\Omega}), \quad 0<p(x), \quad 0\le q(x), \quad x\in\Omega, \tag{1.3}$$

and the liner differential operator on the time is defined as

$$P({}_0^C D_t)={}_0^C D_t^{\alpha}+\sum_{j=1}^{m} r_j {}_0^C D_t^{\alpha_j}, \tag{1.4}$$

where α denotes the principal fractional order, and $\alpha_1,\alpha_2,\cdots,\alpha_m$ are the multi-term fractional orders of the derivatives, which satisfy the condition:

$$0<\alpha_m<\alpha_{m-1}<\cdots<\alpha_1<\alpha<1, \tag{1.5}$$

and $r_1,r_2,\cdots,r_m$ are positive constants; and all of the time-fractional derivatives in (1.4) are defined in the meaning of Caputo from the left-hand side, which for example for $\alpha\in(0,1)$ the fractional derivative of a function $h=h(t)$ at $t\in(0,T]$ is given as

$${}_0^C D_t^{\alpha}h(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{h'(s)}{(t-s)^{\alpha}}ds, \tag{1.6}$$

where $\Gamma(\cdot)$ is the usual Gamma function. See, e.g., Podlubny [34] and Kilbas et al. [16] for the definition and properties of the Caputo's derivative.

From the past few years, there are quite a few researches on the forward problem of multi-term time-fractional diffusion equations like Eq. (1.1), for instance, see [27] for the maximum principle, see [22,28,29] for the uniqueness and existence of the solution, see [8, 11] for the analytic solution, and see [18, 24] for numerical solutions of finite difference method, and see [12] for the finite element solution, etc.

However, for real problems, the part of the boundary, or the initial data, or the diffusion coefficient, or the source term can not be obtained directly and we have to determine them by some additional measurements, which yields to inverse problems arising