Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 6, pp. 1312-1329

DOI: 10.4208/aamm.OA-2016-0185
December 2017

# The Plane Waves Method for Numerical Boundary Identification

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Received 6 November 2016; Accepted (in revised version) 8 May 2017

**Abstract.** We study the numerical identification of an unknown portion of the boundary on which either the Dirichlet or the Neumann condition is provided from the knowledge of Cauchy data on the remaining, accessible and known part of the boundary of a two-dimensional domain, for problems governed by Helmholtz-type equations. This inverse geometric problem is solved using the plane waves method (PWM) in conjunction with the Tikhonov regularization method. The value for the regularization parameter is chosen according to Hansen's L-curve criterion. The stability, convergence, accuracy and efficiency of the proposed method are investigated by considering several examples.

**AMS subject classifications**: 65N35, 65N21, 65N38

**Key words**: Plane waves method, collocation, inverse problem, regularization.

## 1 Introduction

The Helmholtz and modified Helmholtz equations are related to various physical applications in science and engineering. More specifically, these equations are used to describe the Debye-Hückel equation [15], the scattering of a wave [17], the linearization of the Boltzmann equation [35], the vibration of a structure [6], the acoustic cavity problem [12], the radiation wave [19] and the steady-state heat conduction in fins [33]. In

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general, we assume the knowledge of the geometry of the domain of interest, the boundary conditions on the entire boundary of the solution domain and the so-called wave parameter, $\kappa$, and this gives rise to *direct/forward problems* for Helmholtz-type equations, which have been extensively studied both mathematically and numerically, e.g., [24,34]. When one or more of the above conditions for solving the direct problem associated with Helmholtz-type equations are partially or entirely unknown, then an *inverse problem* may be formulated to determine the unknowns from additional responses.

Traditional numerical methods, in conjunction with an appropriately chosen regularization/stabilization method, have been employed to solve inverse problems associated with Helmholtz-type equations, such as the finite-difference method (FDM) [4,5], the finite element method (FEM) [25,26] and the boundary element method (BEM) [39,40], respectively. Both the FDM and the FEM require the discretization of the domain of interest which is time consuming and tedious, especially for complicated geometries. On the other hand, while the BEM is a boundary discretization method and hence reduces the dimensionality of the problem by one, however it requires the evaluation of singular integrals involving the fundamental solution and its normal derivative and the corresponding BEM matrices are fully populated.

An alternative to these traditional numerical methods are the so-called *meshless methods* which have been used extensively in the last two decades for retrieving accurate, stable and convergent numerical solutions to inverse problems for Helmholtz-type equations. The advantages of meshless methods are the ease with which they can be implemented, in particular for problems in complex geometries, their low computational cost and the fact that, in general, they are exempted from integrations that may become cumbersome, especially in three dimensions. Such methods include the boundary particle method (BPM) [13], the singular boundary method (SBM) [14], the method of fundamental solutions (MFS) [16], the boundary knot method (BKM) [23], Kansa's method [28], etc.

The plane waves method (PWM) is a meshless Trefftz method applicable to the solution of boundary value problems governed by the Helmholtz or modified Helmholtz equation, [1,2,44], see also [20, Section 11.1.3]. The PWM has since been applied to the modified Helmholtz equation in [36], for the calculation of the eigenfrequencies of the Laplace operator in [3] and for the solution of inverse problems of Cauchy type in [22]. More recently, it was applied to the solution of direct axisymmetric Helmholtz problems in [29].

The PWM is closely related to another meshless Trefftz method, the method of fundamental solutions (MFS) [16] which has in recent years become very popular for the solution of inverse problems [31,32]. The reason for this popularity is due to the fact that it is meshless and of boundary type, hence the MFS is easy to implement for problems in complex geometries in two and three dimensions. These properties are shared by the PWM which was shown to be an asymptotic version of the MFS in [2]. Moreover, the PWM has a considerable advantage over the MFS as it does not require an external pseudo-boundary on which the sources are to be placed. The location of this pseudo-