Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. 8, No. 6, pp. 1084-1098

DOI: 10.4208/aamm.2015.m1248
December 2016

Bifurcations and Single Peak Solitary Wave Solutions of an Integrable Nonlinear Wave Equation

Wei Wang, Chunhai Li* and Wenjing Zhu

School of Mathematics and Computing Science, Guilin University of Electronic Technology, Jinji Road No. 1, Guilin 541004, China

Received 6 August 2015; Accepted (in revised version) 2 December 2015

Abstract. Dynamical system theory is applied to the integrable nonlinear wave equation $u_t \pm (u^3 - u^2)_x + (u^3)_{xxx} = 0$. We obtain the single peak solitary wave solutions and compacton solutions of the equation. Regular compacton solution of the equation correspond to the case of wave speed $c=0$. In the case of $c \neq 0$, we find smooth soliton solutions. The influence of parameters of the traveling wave solutions is explored by using the phase portrait analytical technique. Asymptotic analysis and numerical simulations are provided for these soliton solutions of the nonlinear wave equation.

AMS subject classifications: 35Q51, 35Q53

Key words: Bifurcation, solitary wave, compaction.

1 Introduction

It is well known that the study of nonlinear wave equations and their solutions are of great importance in many areas of physics. Travelling wave solution is an important type of solution for the nonlinear partial differential equation. Finding their traveling wave solutions of these equations has become a hot research topic for many scholars. Many methods have been used to investigate these types of equations, such as tanh-sech method [1], Lie group method [2], exp-function method, bifurcation method [3–9] and sine-cosine method.

Classically, the solitary wave solutions of nonlinear evolution equations are determined by analytic formulae and serve as prototypical solutions that model physical localized waves. For integrable systems, the solitary waves interact clearly, and are known as solitons. The appearance of non-analytic solitary wave solutions to new classes of nonlinear wave equations, including peakons [10–14], which have a corner at their crest,

*Corresponding author.
Email: chunhai2001@163.com (C. H. Li)

cuspons [11], having a cusped crest and compactons [15–26], which have compact support, has vastly increased the menagerie of solutions appearing in model equations, both integrable and non-integrable.

There are two important nonlinearly dispersive equations. One is the well-known Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + u_{xxx}, \tag{1.1}$$

which was proposed by Camassa and Holm [10] as a model equation for unidirectional nonlinear dispersive waves in shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties. The Camassa-Holm equation have been found to has peakons, cuspons and composite wave solutions [11]. The other is the $K(m,n)$ equation

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \tag{1.2}$$

which was discovered by Rosenau and Hyman [27], where a is a constant and both the convection term $(u^m)_x$ and the dispersion effect term $(u^n)_{xxx}$ are nonlinear. These equations arise in the process of understanding the role of nonlinear dispersion in the formation of structures like liquid drops. Rosenau and Hyman derived solutions called compactons for Eq. (1.2). Xu and Tian [28] introduced the osmosis $K(2,2)$ equation

$$u_t + (u^2)_x - (u^2)_{xxx} = 0, \tag{1.3}$$

where the negative coefficient of dispersion term denotes the contracting dispersion.

In the present work, we consider the following integrable nonlinear wave equation

$$u_t + a(u^3 - u^2)_x + (u^3)_{xxx} = 0, \tag{1.4}$$

where $a = \pm 1$. It's a simple model used for cubic dispersion of presentation. In [29], Rosenau had studied the impact of a non-convex convection on formation of compactons by using this model. Note that whereas the $K(3,3)$ has four local conserved quantities [27]: $\int u dx$, $\int u^4 dx$, $\int u \cos x dx$ and $\int u \sin x dx$, Eq. (1.4) inherits from $K(3,3)$ only two conserved quantities: $\int u dx$, $\int u^4 dx$.

Here, by using bifurcation theory of dynamical system, we consider bifurcation problem of the single peak solitary wave solutions and compacton solutions for the Eq. (1.4).

We look for travelling wave solutions of Eq. (1.4) in the form of $u(x,t) = u(\xi)$ with $\xi = x - ct$, where c is the wave speed. Substituting the traveling wave solution $u(x,t) = u(x-ct)$ into Eq. (1.4), we have the following equation:

$$-cu_\xi + a(u^3 - u^2)_\xi + (u^3)_{\xi\xi\xi} = 0. \tag{1.5}$$

Integrating (1.5) once and setting the integration constant as g, we have

$$-cu + a(u^3 - u^2) + (u^3)_{\xi\xi} = g. \tag{1.6}$$