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Error Estimates of Mixed Methods for Optimal Control Problems Governed by General Elliptic Equations

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Abstract. In this paper, we investigate the error estimates of mixed finite element methods for optimal control problems governed by general elliptic equations. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. We derive L^2 and H^{-1}-error estimates both for the control variable and the state variables. Finally, a numerical example is given to demonstrate the theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: General elliptic equations, optimal control problems, superconvergence, error estimates, mixed finite element methods.

1 Introduction

Optimal control problems have been widely met in all kinds of practical problems. It have been widely studied and applied in the science and engineering numerical simulation. The finite element method was undoubtedly the most widely used numerical method in computing optimal control problems. There have been extensive studies in convergence of the finite element approximation of optimal control problems. For the studies about convergence and superconvergence of finite element approximations for optimal control problems, see, for example, [1,5,9–11,13,15–19,21,22]. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, for example, [7, 14].

However, compared with standard finite element methods, the mixed finite element methods have many advantages. When the objective functional contains gradient of the state variable, we will firstly choose the mixed finite element methods. Chen et al. have

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done some works on a priori error estimates and superconvergence properties of mixed finite elements for optimal control problems, see, for example, [3,4,6]. In [4], Chen used the postprocessing projection operator, which was defined by Meyer and Rösch (see [15]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, Chen et al. derived error estimates and superconvergence of mixed methods for convex optimal control problems in [6]. However, in [6], the authors did not derived a H^{-1}-error estimates for the control variable and the state variables.

The goal of this paper is to derive the error estimates of mixed finite element approximation for an elliptic control problem. Firstly, by use of the duality argument, we derive the superconvergence property between average L^2 projection and the approximation of the scalar function, the convergence order is $h^{\frac{3}{2}}$ as that obtained in [6], which can be seen as a special case of this paper. Then, based on these superconvergence results, we derive L^2 and H^{-1}-error estimates for the optimal control problems. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results.

We consider the following linear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with a pointwise control constraint:

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(a\nabla y+\boldsymbol{b}y)+cy=u, \quad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p}+cy=u, \quad \boldsymbol{p}=-(a\nabla y+\boldsymbol{b}y), \quad x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y=0, \quad x\in\partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad}=\{u\in L^2(\Omega):\ u\geq 0,\ a.e.\ \text{in}\ \Omega\}. \tag{1.5}$$

Moreover, we assume that $0<a_0\leq a\leq a^0$, $a\in W^{1,\infty}(\Omega)$, $0<c\in W^{1,\infty}(\Omega)$, $\boldsymbol{b}\in(W^{1,\infty}(\Omega))^2$, $y_d\in H^1(\Omega)$, $\boldsymbol{p}_d\in(H^1(\Omega))^2$, and ν is a fixed positive number. We also assume that the following condition holds [8]:

$$\boldsymbol{b}^2\leq 4(1-\gamma)ac \ \text{ for some } \ \gamma\in(0,1). \tag{1.6}$$

The plan of this paper is as follows. In Section 2, we construct the mixed finite element approximation scheme for elliptic optimal control problem (1.1)-(1.4) and give its