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Explorations and Expectations of Equidistribution Adaptations for Nonlinear Quenching Problems

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Dedicated to Professor Graeme Fairweather on the occasion of his 70^{th} birthday.

Abstract. Finite difference computations that involve spatial adaptation commonly employ an equidistribution principle. In these cases, a new mesh is constructed such that a given monitor function is equidistributed in some sense. Typical choices of the monitor function involve the solution or one of its many derivatives. This straightforward concept has proven to be extremely effective and practical. However, selections of core monitoring functions are often challenging and crucial to the computational success. This paper concerns six different designs of the monitoring function that targets a highly nonlinear partial differential equation that exhibits both quenching-type and degeneracy singularities. While the first four monitoring strategies are within the so-called *primitive* regime, the rest belong to a later category of the *modified* type, which requires the priori knowledge of certain important quenching solution characteristics. Simulated examples are given to illustrate our study and conclusions.

AMS subject classifications: 65K20, 65M50, 35K65

Key words: Degeneracy, quenching singularity, adaptive difference method, arc-length, monitoring function, splitting method.

1 Introduction

Temporal and spatial adaptations have been playing an important role for computing the numerical solution of singular or near singular differential equations. Commonly, adaptations stem from the equidistribution of a particular monitor function [3]. For singular problems, the appropriate choice of a monitor function is not clear, as say for blow-up problems where the monitor function is chosen to minimize the local truncation error. Still the ultimate goal of the employed strategy is to optimize discretization steps for

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matching key physical properties of solutions, or easing the domain geometric sophistication. Adaptive mechanisms are often achieved through monitoring closely the most sensitive features of the multi-physical system anticipated, such as the velocity of fluids, location of wave fronts, and evidence of potential singularities. In this paper, through a frequently used two-dimensional reaction-diffusion equation of the quenching type, we discuss several effective adaptation designs. The exploration brings to the surface subtle issues in quenching computations and offers a cautionary reminder that a particularly tailored adaptation must be carefully screened prior to employment.

Let Ω be an open unit square. A typical two-dimensional degenerate quenching model can be comprised as

$$\phi(x,y)u_t=\alpha u_{xx}+\beta u_{yy}+f(u), \qquad (x,y)\in\Omega, \qquad t>0, \tag{1.1a}$$

$$u(x,y,t)=0, \qquad (x,y)\in\Gamma, \qquad t>0, \tag{1.1b}$$

$$u(x,y,0)=u_0(x,y), \qquad (x,y)\in\Omega, \tag{1.1c}$$

where Γ is the boundary of Ω, $\alpha\geq\beta>0$ are constants, and $\phi(x,y)=\phi(y,x)>0$, $(x,y)\in\Omega$. A degeneracy occurs if ϕ diminishes at certain points on Γ. The source term f is highly nonlinear, positive, and approaches infinity as $u\to 1^-$. We adopt the standard nomenclature for quenching, first proposed by [7], that is, the solution u is said to *quench* if the time derivative u_t becomes unbounded in finite time. That time is called the *quenching time*. As discussed in [1,5], a single point quenching singularity of (1.1a)-(1.1c), if occurs, must locate on the line segment of $y=x$, $0<x<1$. An interesting nuance of higher dimensional quenching problems is that depending on the size *and* shape of the domain the solution may or may not quench. Calculating *critical quenching domains* and times has been a primary focus of numerical and theoretical analysis [1,4–6,8,11,12,14–17]. The numerical approaches have contained a mix of uniform and nonuniform grids often employing temporal adaptation solely. It still remains to be seen how to best adapt the spatial grid to improve on the overall numerical accuracy, efficiency, and robustness, let alone the effects of such adaptations on the computation itself.

In quenching phenomena, it has been shown that if a solution quenches at a finite value, then the rate of change function, that is, the temporal or spatial derivative, blows up faster than an exponential rate [5,13]. This leads to the following procedure for temporal adaptation based on the equidistribution of u_t. We adopt the implicit equation for a new time step in each advancement,

$$\big(u'_{k+1}-u'_k\big)^2\cdot e_j+\big(\tau_k^{(i)}\big)^2=\big(u'_k-u'_{k-1}\big)^2\cdot e_j+\tau_{k-1}^2, \tag{1.2}$$

where u_k, u_{k+1} are numerical solutions at temporal levels k and $k+1$, respectively, and $e_j\in\mathbb{R}^N$ is the jth unit vector, $1\leq j\leq N$. The notation $(u)^p$ means that each of the vector's components is raised to the power p, and the initial step τ_0 is given. The updated temporal step τ_k is taken to be the minimum, that is,

$$\tau_k=\min_{1\leq j\leq N}\tau_k^{(j)}.$$