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Mixed Finite Element Methods for Elastodynamics Problems in the Symmetric Formulation

Yan Yang[1,2] and Shiquan Zhang[1,*]

[1] *School of Mathematics, Sichuan University, Chengdu 610064, China*
[2] *School of Sciences, Southwest Petroleum University, Chengdu 610500, China*

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Abstract. In this paper, we analyze semi-discrete and fully discrete mixed finite element methods for linear elastodynamics problems in the symmetric formulation. For a large class of conforming mixed finite element methods, the error estimates for each scheme are derived, including the energy norm and L^2 norm for stress, and the L^2 norm for velocity. All the error estimates are robust for the nearly incompressible materials, in the sense that the constant bound and convergence order are independent of Lamé constant λ. The stress approximation in both norms, as well as the velocity approximation in L^2 norm, are with optimal convergence order. Finally numerical experiments are provided to confirm the theoretical analysis.

AMS subject classifications: 65N15, 65N30, 74H15, 74S05

Key words: Mixed finite element, elastodynamics, symmetric formulation, robust error estimates.

1 Introduction

Let $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ be a bounded domain with Lipschitz continuous boundary $\Gamma=\bar{\Gamma}_D\cup\bar{\Gamma}_N$, where $\Gamma_D\cap\Gamma_N=\emptyset$ and $\text{meas}(\Gamma_D)>0$. We consider the following linear elastodynamics problem:

$$\begin{cases} \rho\dfrac{\partial^2\boldsymbol{u}}{\partial t^2}-\mathbf{div}\sigma=\boldsymbol{f} & \text{in } \Omega\times[0,T],\\ \sigma=2\mu\varepsilon(\boldsymbol{u})+\lambda\mathbf{div}\boldsymbol{u}\boldsymbol{I} & \text{in } \Omega\times[0,T],\\ \boldsymbol{u}=\boldsymbol{g} & \text{on } \Gamma_D\times[0,T],\\ \sigma\boldsymbol{n}=\boldsymbol{0} & \text{on } \Gamma_N\times[0,T],\\ \boldsymbol{u}=\boldsymbol{u}_0,\quad \dfrac{\partial\boldsymbol{u}}{\partial t}=\boldsymbol{v}_0 & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

*Corresponding author.
Emails: yyan2011@163.com (Y. Yang), shiquanzhang@scu.edu.cn (S. Q. Zhang)

where $\rho=\rho(\boldsymbol{x})$ is the density of the medium satisfying $0<\rho_0\le\rho\le\rho_1<\infty$, $\boldsymbol{u}(\boldsymbol{x},t)\in\mathbb{R}^d$ the displacement vector, $\boldsymbol{\sigma}(\boldsymbol{x},t)\in\mathbb{R}^{d\times d}$ the symmetric stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u}):=\frac{\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T}{2}$ the strain tensor, $\lambda>0$ and $\mu>0$ the Lamé coefficients, and $\boldsymbol{I}$ the $d\times d$ identity matrix. $T>0$ denotes the final time, $\boldsymbol{f}(\boldsymbol{x},t)\in\mathbb{R}^d$ the body force, $\boldsymbol{g}(\boldsymbol{x},t)\in\mathbb{R}^d$ the prescribed boundary displacement on Γ_D, $\boldsymbol{n}$ the unit outward vector normal to Γ, $\boldsymbol{u}_0(\boldsymbol{x})$ and $\boldsymbol{v}_0(\boldsymbol{x})$ the initial displacement and velocity data, respectively.

Besides, the initial data must satisfy the compatibility conditions

$$\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\boldsymbol{n}|_{\Gamma_N}=\boldsymbol{0},\quad \boldsymbol{u}_0|_{\Gamma_D}=\boldsymbol{g}(0). \tag{1.2}$$

From the second equation of (1.1), we can easily get

$$\boldsymbol{\varepsilon}(\boldsymbol{u})=\frac{1}{2\mu}\left(\boldsymbol{\sigma}-\frac{\lambda}{2\mu+d\lambda}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right)=:\mathcal{A}\boldsymbol{\sigma}.$$

Then, if $\frac{\partial\boldsymbol{g}}{\partial t}$ exists, by introducing the velocity field variable $\boldsymbol{v}=\frac{\partial\boldsymbol{u}}{\partial t}$, we can rewrite the displacement-stress fields model (1.1) into the following first-order velocity-stress fields system:

$$\begin{cases}\rho\dfrac{\partial\boldsymbol{v}}{\partial t}-\mathbf{div}\boldsymbol{\sigma}=\boldsymbol{f} & \text{in } \Omega\times[0,T],\\ \mathcal{A}\dfrac{\partial\boldsymbol{\sigma}}{\partial t}-\boldsymbol{\varepsilon}(\boldsymbol{v})=\boldsymbol{0} & \text{in } \Omega\times[0,T],\\ \boldsymbol{v}=\dfrac{\partial\boldsymbol{g}}{\partial t} & \text{on } \Gamma_D\times[0,T],\\ \boldsymbol{\sigma}\boldsymbol{n}=\boldsymbol{0} & \text{on } \Gamma_N\times[0,T],\end{cases} \tag{1.3}$$

with initial conditions

$$\boldsymbol{\sigma}(\cdot,0)=\boldsymbol{\sigma}_0:=\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_0),\quad \boldsymbol{v}(\cdot,0)=\boldsymbol{v}_0. \tag{1.4}$$

In the displacement-stress formulation (1.1), it retains the same variables as for elastostatics. However, the constitutive equation does not involve time differentiation, which leads to a system of differential-algebraic equations in time. Owing to such character, it needs special design to obtain stable time stepping methods for fully discrete schemes. To overcome such difficulty, Hughes etc. [23] proposed the space-time finite element methods which combine the using of discontinuous Galerkin method in time and stabilizing terms of least-squares type. In [25], a C^0-continuous time stepping displacement-type finite element method is constructed to obtain stable time discretization. Boulaajine etc. [10, 11] proposed dual mixed finite element methods with explicit/implicit Newmark schemes for the time approximation. We refer to [30,31] for semi-discrete and fully discrete of hybrid stress finite element methods for elastodynamics, where a second-order center difference and an implicit second-order difference were used for the time discretization, respectively.

In the velocity-stress scheme [16, 26], the displacement is not a primary unknown, but can be recoverable as the time-integral of the velocity. The most important advantage of this scheme is that it leads to a standard hyperbolic system, thus many stable