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Fully Finite Element Adaptive AMG Method for Time-Space Caputo-Riesz Fractional Diffusion Equations

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Abstract. The paper aims to establish a fully discrete finite element (FE) scheme and provide cost-effective solutions for one-dimensional time-space Caputo-Riesz fractional diffusion equations on a bounded domain Ω. Firstly, we construct a fully discrete scheme of the linear FE method in both temporal and spatial directions, derive many characterizations on the coefficient matrix and numerically verify that the fully discrete FE approximation possesses the saturation error order under $L^2(\Omega)$ norm. Secondly, we theoretically prove the estimation $1+\mathcal{O}(\tau^\alpha h^{-2\beta})$ on the condition number of the coefficient matrix, in which τ and h respectively denote time and space step sizes. Finally, on the grounds of the estimation and fast Fourier transform, we develop and analyze an adaptive algebraic multigrid (AMG) method with low algorithmic complexity, reveal a reference formula to measure the strength-of-connection tolerance which severely affect the robustness of AMG methods in handling fractional diffusion equations, and illustrate the well robustness and high efficiency of the proposed algorithm compared with the classical AMG, conjugate gradient and Jacobi iterative methods.

AMS subject classifications: 35R11, 65F10, 65F15, 65N55

Key words: Caputo-Riesz fractional diffusion equation, fully discrete space-time FE scheme, condition number estimation, algorithmic complexity, adaptive AMG method.

1 Introduction

In recent years, there has been an explosion of research interest in numerical solutions for fractional differential equations, mainly due to the following two aspects: (i) the huge

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majority can't be solved analytically, (ii) the analytical solution (if luckily derived) always involves certain infinite series which sharply drives up the costs of its evaluation. Numerous numerical methods have been proposed to approximate more accurately and faster, such as finite difference (FD) method [1–11], finite element (FE) method [12–20], finite volume method [21] and spectral (element) method [22–27]. An essential challenge against standard differential equations lies in the presence of the fractional differential operator, which gives rise to nonlocality (space fractional, nearly dense or full coefficient matrix) or memory-requirement (time fractional, the entire time history of evaluations) issue, resulting in a vast computational cost.

Preconditioned Krylov subspace methods are regarded as one of potential solutions to the aforementioned challenge. Numerous preconditioners with various Krylov subspace methods have been constructed for one- and two-dimensional, linear and nonlinear space-fractional diffusion equations (SFDE) [28–31]. Multigrid method has been proven to be a superior solver and preconditioner for ill-conditioned Toeplitz systems as well as SFDE. Pang and Sun proposed an efficient and robust geometric multigrid (GMG) with fast Fourier transform (FFT) for one-dimensional SFDE by an implicit FD scheme [32]. Bu et al. employed this GMG to multi-term time-fractional advection-diffusion equations via a fully discrete scheme by FD method in temporal and FE method in spatial direction [33]. Jiang and Xu constructed optimal GMG for two-dimensional SFDE to get FE approximations [34]. Chen et al. made the first attempt to present an algebraic multigrid (AMG) method with line smoothers to the fractional Laplacian through localizing it into a nonuniform elliptic equation [35]. Zhao et al. invoked GMG for one-dimensional Riesz SFDE by an adaptive FE scheme using hierarchical matrices [36]. Chen and Deng exploited GMG's coarsening strategy and grid-transfer operators, equipped with Galerkin coarse-grid operator to produce a robust multigrid for nonlocal models in a finite range of interactions [37]. From the survey of references, despite quite a number of contributions to numerical methods and preconditioners, there are no calculations taking into account of fully discrete FE schemes and AMG methods for time-space Caputo-Riesz fractional diffusion equations.

In this paper, we are concerned with the following time-space Caputo-Riesz fractional diffusion equation (CR-FDE)

$$
\begin{aligned}
&{}_0^C D_t^\alpha u(x,t) = \frac{\partial^{2\beta} u(x,t)}{\partial |x|^{2\beta}} + f(x,t), && t\in I=(0,T], \quad x\in\Omega=(a,b), && \text{(1.1a)}\\
&u(x,t)=0, && t\in I, \quad x\in\partial\Omega, && \text{(1.1b)}\\
&u(x,0)=\psi_0(x), && x\in\Omega, && \text{(1.1c)}
\end{aligned}
$$

with orders $\alpha\in(0,1)$ and $\beta\in(1/2,1)$, the Caputo and Riesz fractional derivatives are respectively defined by

$$
{}_0^C D_t^\alpha u = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial u}{\partial s}ds, \quad \frac{\partial^{2\beta}u}{\partial|x|^{2\beta}} = -\frac{1}{2\cos(\beta\pi)}\left({}_x D_L^{2\beta}u + {}_x D_R^{2\beta}u\right),
$$