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Structure-Preserving Wavelet Algorithms for the Nonlinear Dirac Model

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Abstract. The nonlinear Dirac equation is an important model in quantum physics with a set of conservation laws and a multi-symplectic formulation. In this paper, we propose energy-preserving and multi-symplectic wavelet algorithms for this model. Meanwhile, we evidently improve the efficiency of these algorithms in computations via splitting technique and explicit strategy. Numerical experiments are conducted during long-term simulations to show the excellent performances of the proposed algorithms and verify our theoretical analysis.

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Key words: Structure-preserving, wavelet collocation, conservation laws, nonlinear Dirac equation.

1 Introduction

In this paper, we consider the nonlinear Dirac (NLD) model

$$\begin{cases} \Psi_t = A\Psi_x + \mathrm{i}f(|\Psi_1|^2 - |\Psi_2|^2)B\Psi, \\ \Psi_1(x,0) = \phi_1(x), \quad \Psi_2(x,0) = \phi_2(x), \end{cases} \tag{1.1}$$

where $\Psi = (\Psi_1, \Psi_2)^{\mathrm{T}}$ is a spinorial wave function of a particle with the spin-$\frac{1}{2}$; Ψ_1 and Ψ_2 are complex functions describing the up and down states of the spin-$\frac{1}{2}$ particle, respectively; $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, $f(s)$ is a real function of a real variable s, A and B

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are matrices

$$A=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Dirac equation, formulated by the British physicist Paul Dirac in 1928, has been playing a fundamental role in various areas of modern physics, especially in description of interacting particles and fields.

In this paper, the charge $\mathcal{Q}$, the energy $\mathcal{E}$ and the linear momentum $\mathcal{M}$ are defined by

$$\begin{aligned}
\mathcal{Q}(\Psi)(t)&=\int_{\mathbf{R}}(|\Psi_1(x,t)|^2+|\Psi_2(x,t)|^2)dx,\\
\mathcal{E}(\Psi)(t)&=\int_{\mathbf{R}}\Big[\Im\Big(\overline{\Psi}_1\frac{\partial}{\partial x}\Psi_2+\overline{\Psi}_2\frac{\partial}{\partial x}\Psi_1\Big)+\widetilde{f}(|\Psi_1|^2-|\Psi_2|^2)\Big]dx,\\
\mathcal{M}(\Psi)(t)&=\int_{\mathbf{R}}\Im\Big(\overline{\Psi}_1\frac{\partial}{\partial x}\Psi_1+\overline{\Psi}_2\frac{\partial}{\partial x}\Psi_2\Big)dx,
\end{aligned}$$

where $\Im(\Psi)$ and $\overline{\Psi}$ stand for the imaginary part and the conjugate of the complex quantity Ψ, and $\widetilde{f}$ is a primitive function of f.

The NLD equation (1.1) admits the following conservation laws [1]:

Proposition 1.1. If the solution Ψ satisfies

$$\lim_{|x|\to+\infty}|\Psi(x,t)|=0 \quad \text{and} \quad \lim_{|x|\to+\infty}|\partial_x\Psi(x,t)|=0 \quad \text{uniformly for } t\in\mathbf{R},$$

then

$$\frac{d}{dt}\mathcal{Q}(t)=0, \quad \frac{d}{dt}\mathcal{E}(t)=0 \quad \text{and} \quad \frac{d}{dt}\mathcal{M}(t)=0.$$

This paper will focus on an important particular case [1] of the NLD equation (1.1)

$$\begin{cases}
\dfrac{\partial\Psi_1}{\partial t}+\dfrac{\partial\Psi_2}{\partial x}+\mathrm{i}m\Psi_1+2\mathrm{i}\lambda(|\Psi_2|^2-|\Psi_1|^2)\Psi_1=0,\\
\dfrac{\partial\Psi_2}{\partial t}+\dfrac{\partial\Psi_1}{\partial x}-\mathrm{i}m\Psi_2+2\mathrm{i}\lambda(|\Psi_1|^2-|\Psi_2|^2)\Psi_2=0,
\end{cases} \tag{1.2}$$

that is, $f(s)=m-2\lambda s$ in Eq. (1.1), where $m,\lambda\in\mathbf{R}$, λ is the nonlinear parameter.

Given $\Psi_1=q_1+\mathrm{i}q_2$, $\Psi_2=q_3+\mathrm{i}q_4$ with q_k $(k=1,2,3,4)$ being real functions, the NLD equation (1.2) is rewritten as a system of real-value equations

$$\begin{cases}
\dfrac{\partial}{\partial t}q_1+\dfrac{\partial}{\partial x}q_3-mq_2-2\lambda(q_3^2+q_4^2-q_1^2-q_2^2)q_2=0,\\
\dfrac{\partial}{\partial t}q_2+\dfrac{\partial}{\partial x}q_4+mq_1+2\lambda(q_3^2+q_4^2-q_1^2-q_2^2)q_1=0,\\
\dfrac{\partial}{\partial t}q_3+\dfrac{\partial}{\partial x}q_1+mq_4+2\lambda(q_3^2+q_4^2-q_1^2-q_2^2)q_4=0,\\
\dfrac{\partial}{\partial t}q_4+\dfrac{\partial}{\partial x}q_2-mq_3-2\lambda(q_3^2+q_4^2-q_1^2-q_2^2)q_3=0.
\end{cases} \tag{1.3}$$