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The Oseen Type Finite Element Iterative Method for the Stationary Incompressible Magnetohydrodynamics

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Abstract. In this article, by applying the Stokes projection and an orthogonal projection with respect to curl and div operators, some new error estimates of finite element method (FEM) for the stationary incompressible magnetohydrodynamics (MHD) are obtained. To our knowledge, it is the first time to establish the error bounds which are explicitly dependent on the Reynolds numbers, coupling number and mesh size. On the other hand, The uniform stability and convergence of an Oseen type finite element iterative method for MHD with respect to high hydrodynamic Reynolds number R_e and magnetic Reynolds number R_m, or small $\delta=1-\sigma$ with

$$\sigma=\sqrt{2}C_0^2\max\{1,\sqrt{2}S_c\}\|\mathbf{F}\|_{-1}/(\min\{R_e^{-1},S_cC_1R_m^{-1}\})^2$$

(C_0, C_1 are constants depending only on Ω and $\mathbf{F}$ is related to the source terms of equations) are analyzed under the condition that $h\le(\|\mathbf{F}\|_{-1}/\|\mathbf{F}\|_0)^{1/2}\delta$. Finally, some numerical tests are presented to demonstrate the effectiveness of this algorithm.

AMS subject classifications: 35Q30, 65M60, 65N30, 76D05

Key words: Uniform stability, convergence, Oseen type iterative method, finite element method, stationary incompressible magnetohydrodynamics.

1 Introduction

The field of MHD studies the theory of macroscopic interaction between electrically conducting fluids and electromagnetic fields. Many MHD problems consist of a viscous, incompressible fluid which has the property of electric current conduction and interacting

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with electromagnetic fields. There are lots of applications in astronomy and geophysics as well as engineering problems, such as metallurgical engineering, electromagnetic pumping, stirring of liquid metals, liquid metal cooling of nuclear reactors, electromagnetic casting of metals and measuring flow quantities based on induction. See [1,2] for more understanding of the MHD field. In this paper, we consider the steady incompressible flow of an electrically conducting fluid in the presence of a magnetic field, as modelled by the system [3]:

$$R_e^{-1}\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p-S_c\mathrm{curl}\mathbf{B}\times\mathbf{B}=\mathbf{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$S_cR_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B})-S_c\mathrm{curl}(\mathbf{u}\times\mathbf{B})=\mathbf{g} \quad \text{in } \Omega, \tag{1.1b}$$

$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\nabla\cdot\mathbf{B}=0 \quad \text{in } \Omega, \tag{1.1d}$$

along with boundary conditions

$$\mathbf{u}|_{\partial\Omega}=0, \quad (\mathbf{B}\cdot\mathbf{n})|_{\partial\Omega}=0, \quad (\mathbf{n}\times\mathrm{curl}\mathbf{B})|_{\partial\Omega}=0, \tag{1.2}$$

where Ω is a convex polygonal/polyhedral domain in R^d ($d=2$ or 3). This nonlinear multiple variable coupling MHD system is characterized by three equation parameters: hydrodynamic Reynolds number R_e, magnetic Reynolds number R_m and coupling number S_c. Here, $\mathbf{u}$ is the fluid velocity, p the hydrodynamic pressure, $\mathbf{B}$ the magnetic field and $\mathbf{n}$ the unit outward normal on $\partial\Omega$. The functions $\mathbf{f}$ and $\mathbf{g}$ represent external force terms.

A considerable amount of research activity has been devoted to the analysis of numerical methods for the simulation of MHD flows by applying various FEMs. It has been shown by Gunzburger et al. in [3] that the existence and uniqueness of the solution to a weak formulation of the MHD equations can be guaranteed. Schötzau [4], Hasler et al. [5] and Greif et al. [6] have carried out error analysis on their mixed finite element solutions to MHD problems and established optimal error estimates. On the other hand, because the small hydrodynamic diffusion may induce some numerical instabilities, stabilization techniques are applied for the solution to MHD equations. Salah et al. [7] and Codina [8] have used a stabilized FEM for MHD problem by including a magnetic pressure as unknowns to enforce the divergence free condition for the numerical approximation of the magnetic field. Gerbeau [9] has presented a stabilized FEM procedure refers to the velocity, pressure and magnetic field, as well as a convergence proof. What's more, in the papers by Layton et al. [10], Aydın et al. [11], Dong and He [12], they have proposed two-level FEM, based on solving a small nonlinear system on a coarse mesh and a large linear system on a fine mesh, to save computational time without losing the convergence rate. As for the nonstationary MHD equations, the unconditional convergence of the Euler semi-implicit scheme has been studied by He [13] and the full discretization of Crank-Nicolson scheme at small magnetic Reynolds numbers has been investigated by Yuksel and Ingram [14].