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Simulations of Two-Step Maruyama Methods for Nonlinear Stochastic Delay Differential Equations

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Abstract. In this paper, we investigate the numerical performance of a family of P-stable two-step Maruyama schemes in mean-square sense for stochastic differential equations with time delay proposed in [8, 10] for a certain class of nonlinear stochastic delay differential equations with multiplicative white noises. We also test the convergence of one of the schemes for a time-delayed Burgers' equation with an additive white noise. Numerical results show that this family of two-step Maruyama methods exhibit similar stability for nonlinear equations as that for linear equations.

AMS subject classifications: 65C20, 65C30, 65Q20

Key words: P-stability in mean-square sense, two-step Maruyama methods, nonlinear stochastic delay differential system, Burgers' equation.

1 Introduction

We consider numerical schemes for stochastic delay differential equations (SDDEs), which have been increasingly used to model the effects of noise and time delay on various types of complex systems, such as delayed visual feedback systems [5], control problems [14, 24], the dynamics of noisy bi-stable systems with delay [26], etc. SDDEs are also used in modeling diseases [4, 6] and in models of stock markets [15].

Some one-step numerical schemes for SDDEs and their convergence and stability properties have been established recently [3, 13, 17, 20, 27]. Here we focus on stochastic multi-step methods for SDDEs, which can be treated as an nontrivial extension of the multi-step methods of stochastic ordinary differential equations (SODEs), i.e., with no time delay. For an early review of multi-step methods for SODEs, see [16, 22]. Some more recent studies on SODEs can be found in [9] (two-step Maruyama methods), [12]

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(stochastic Adams-Bashforth scheme), [7] (Adams-type schemes), and also [11] (even high order multi-step methods).

Numerical methods for SDDES can also be considered as extension from deterministic delay differential equations (DDEs). A review of numerical methods and their stability for DDEs can be found in [2]. The stability of SDDEs is a bit different from the SODEs, as we may require some conditions on the size of delay and time step size of the numerical methods. For example, P-stability introduced by Barwell [1] refers that the numerical solution of the delay differential equation $y'(t) = ay(t) + by(t-\tau)$ goes to zero when time goes to infinity for any step size h and the time delay $\tau = mh$ (m is an integer), provided that $|b| < -\mathrm{Re}(a)$; see also [25], where Tian and Kuang considered the P-stability of linear multistep methods for DDEs. We will also adopt this concept for the stability of numerical methods for a linear scalar SDDE.

Inspired by [8, 10], we investigate the numerical performance in this paper of a family of two-step Maruyama schemes for a class of the following scalar equation

$$\mathrm{d}X(t) = f\big(t, X(t), X(t-\tau)\big)\mathrm{d}t + g\big(t, X(t), X(t-\tau)\big)\mathrm{d}W(t), \quad t \in J, \tag{1.1a}$$

$$X(t) = \xi(t), \quad t \in [-\tau, 0], \tag{1.1b}$$

where τ is a positive fixed delay, $J = [0, T]$, $W(t)$ is a one-dimensional standard Wiener process and the functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In [8], multi-step methods are proposed for m-dimensional systems of Itô SDDEs with d driving Wiener processes and multiple delay, and their properties are studied concerning consistency, numerical stability and convergence. In [10], a series of conditions of parameters of the two-step Maruyama method for SDDEs are given. Under these conditions the family of two-step Maruyama schemes are proved to be P-stable in mean-square sense for a linearized equation of (1.1) as follows:

$$\mathrm{d}X(t) = [aX(t) + bX(t-\tau)]\mathrm{d}t + [cX(t) + dX(t-\tau)]\mathrm{d}W(t), \quad t \geq 0, \tag{1.2a}$$

$$X(t) = \xi(t), \quad t \in [-\tau, 0], \tag{1.2b}$$

where $a, b, c, d \in \mathbb{R}$, τ is a positive fixed delay, $W(t)$ is a one-dimensional standard Wiener process and $\xi(t)$ is a $\mathrm{C}([-\tau, 0]; \mathbb{R})$-valued initial segment. Here we aim to test the aforementioned stability and convergence of two-step Maruyama methods for some scalar or system of nonlinear stochastic differential equations with time delay.

The paper is outlined as follows. In Section 2, we provide some necessary notations and preliminaries on SDDEs, including some properties of analytical solutions to Eq. (1.2). We also introduce in this section the two-step Maruyama methods and their convergence properties and derive a family of P-stable two-step Maruyama methods in mean square sense under certain conditions. Section 3 illustrates the P-stability of these two-step Maruyama methods with numerical examples for a nonlinear delay equation with multiplicative white noises and a stochastic delay differential system. Before we conclude, we compute a Burgers' equation with time delay and additive white noise by some of the proposed schemes and show the mean-square convergence of the scheme.