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A Modified Helmholtz Equation with Impedance Boundary Conditions

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Abstract. Here considered is the problem of transient electromagnetic scattering from overfilled cavities embedded in an impedance ground plane. An artificial boundary condition is introduced on a semicircle enclosing the cavity that couples the fields from the infinite exterior domain to those fields inside. A Green's function solution is obtained for the exterior domain, while the interior problem is solved using finite element method. Well-posedness of the associated variational formulation is achieved and convergence and stability of the numerical scheme confirmed. Numerical experiments show the accuracy and robustness of the method.

AMS subject classifications: 65N30, 65N15

Key words: Helmholtz equation, impedance boundary conditions, finite element method.

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1 Introduction

The phenomenon of electromagnetic scattering by cavity-backed apertures has been an area of intense research in recent years. There is a wealth of results reported in both the engineering literature (see, for example, [1–4]) and mathematical journals (see [5–8] and the references therein). It is a common simplifying assumption that the cavity opening coincides with the aperture on a perfect electric conducting (PEC) ground plane. For over-filled cavities we further mention the works [9–13]. We note that most of the published work deals with either cavities with PEC ground planes or time-harmonic problems. The only mathematical treatment of transient problem with overfilled cavities appears to be reported in [9]. We are not aware of any work in that framework, transient and overfilled, under the more prevalent impedance boundary condition (IBC). The aim of this paper is to fill this gap by extending the results of [9] for PEC boundary conditions to IBC. Specifically we develop a hybrid integral

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equation/finite element method that is mathematically wellposed and numerically robust. This model is clearly more applicable physically, yet mathematically more challenging. Specifically, the usual separation of variables approach associated with PEC boundary conditions, or Dirchlet and Newmann boundary conditions, are no longer valid. Our key approach is the development of Green's functions that serve as solutions for the infinite exterior domain. We also, for the first time, numerically implement the method under mixed boundary conditions.

The paper is organized as follows. In Section 2, we establish the mathematical formulation of the problem. Section 3 focuses on the exterior problem where Green's function is derived and boundary operator analyzed. Variational formulation is developed and proved wellposed in Section 4. The paper is concluded in Section 5 with results from some of our numerical experiments.

2 Mathematical formulation

Let $\Omega \subset \mathbb{R}^2$ be the cross-section of a z-invariant cavity in the infinite ground plane, such that its fillings, with material of relative permittivity $\varepsilon_r \geq 1$, protrude above the ground plane. We denote S as the cavity wall and Γ the cavity aperture so that $\partial\Omega = S \cup \Gamma$. The infinite ground plane excluding the cavity opening is denoted as Γ_{ext} and the infinite homogenous, isotropic region above the cavity as $\mathcal{U} = \mathbb{R}^2_+ \setminus \Omega$. Furthermore, let $\mathcal{B}_R$ be a semicircle of radius R, centered at the origin and surrounded by free space, large enough to completely enclose the overfilled portion of the cavity. We denote the region bounded by $\mathcal{B}_R$ and the cavity wall S as Ω_R, so that Ω_R consists of the cavity itself and the homogeneous part between $\mathcal{B}_R$ and Γ. Let $\mathcal{U}_R$ be the homogeneous region outside of Ω_R; that is, $\mathcal{U}_R = \{(r,\theta) : r > R,\ \ 0 < \theta < \pi\}$. Refer to Fig. 1 for the complete problem geometry.

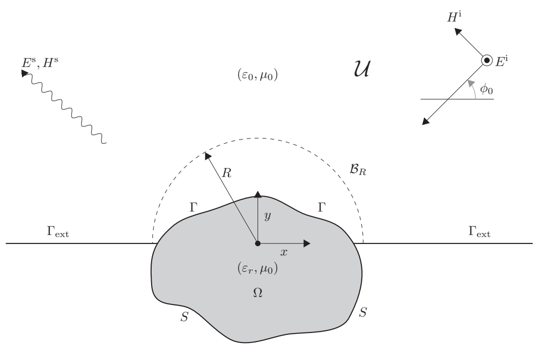

Figure 1: Problem Geometry-TM polarization depicted.

Giving the incident fields $(\boldsymbol{E}^{\rm i}, \boldsymbol{H}^{\rm i})$ impinging on the overfilled cavity, we wish to determine the resulting scattered fields $(\boldsymbol{E}^{\rm s}, \boldsymbol{H}^{\rm s})$. Due to the uniformity in the z-axis, the fields can be decomposed into two fundamental polarizations: transverse magnetic (TM) and transverse electric (TE). Here, for demonstration, we analyze the TM