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The Eulerian-Lagrangian Method with Accurate Numerical Integration

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Abstract. This paper is devoted to the study of the Eulerian-Lagrangian method (ELM) for convection-diffusion equations on unstructured grids with or without accurate numerical integration. We first propose an efficient and accurate algorithm to calculate the integrals in the Eulerian-Lagrangian method. Our approach is based on an algorithm for finding the intersection of two non-matching grids. It has optimal algorithmic complexity and runs fast enough to make time-dependent velocity fields feasible. The evaluation of the integrals leads to increased precision and the unconditional stability. We demonstrate by numerical examples that the ELM with our proposed algorithm for accurate numerical integration has the following two features: first it is much more accurate and more stable than the ones with traditional numerical integration techniques and secondly the overall cost of the proposed method is comparable with the traditional ones.

AMS subject classifications: 65D18, 65D30, 65M25, 65N30

Key words: Eulerian-Lagrangian method, intersection of non-matching grids, exact integration.

1 Introduction

The Eulerian-Lagrangian Method (ELM), also called Semi-Lagrangian Method (SLM), is known as an efficient numerical method for both pure convection and convection-diffusion problems. It works by interpreting the convection term and the time derivative as a material derivative. An implicit discretization based on this approach results in an unconditionally stable scheme that leads to a symmetric positive definite system. The price is a more complicated implementation. In particular, it is noted that inner products of test functions with the solution of the previous time step propagated along

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the integral lines of the velocity field. However, the inner products generally can not be evaluated exactly.

The convergence analysis for the Eulerian-Lagrangian method has been established in [8] for the pure convection problem, and in [2,9] for the convection-diffusion problem. In these works it is assumed that the underlying integrals are evaluated exactly. As demonstrated in [7], ELM may not converge if numerical integration is not accurate enough. Indeed, both theoretical and numerical results show that the numerical approximation of the integration may lead to numerical instability, unless quadrature rules of sufficiently high order are used. Finite element interpolation has been often used for numerical integration, but this approach is known to introduce too much numerical diffusion [5,6,11].

In the implementation of ELM method, inner products that involve two finite element functions on two different grids need to be evaluated. How to accurately evaluate these integration accurately is the main topic of this paper. For two dimensional rectangular grids, an approach called ”area weighting” was introduced in [7]. For two dimensional triangular grids, a special algorithm for ”exact projection” was introduced in [10]. Attaining the information needed by integration is still a time-consuming task and the algorithm to find the intersection of two non-matching grids is output-sensitive or intersection-sensitive (see [1]).

In this paper we introduce a more general and more efficient algorithm to evaluate the inner products. We followed and improved an algorithm proposed by [4] (which was designed for mortar finite element computations) to find the intersection of two non-matching grids. This type of algorithms work for conforming grids of any space dimensions and for all element types and they appear to be more general and more effective than the earlier methods proposed in the literature (e.g., [3] designed for the moving mesh method).

Using the accurate integration method mentioned above, we will use numerical examples to show the resulting ELM can achieve the optimal convergence rate and we will also demonstrate the accurate integration will improve the precision of the numerical solution with only marginal extra computational complexity and such an improvement is more significant for solution with singularities.

The rest of this paper is organized as follows. In Section 2, we introduce ELM for convection-diffusion equations. In Section 3, we describe a specific algorithm of accurate integration with details of the algorithm for finding the intersection of two non-matching grids in arbitrary dimensional case. We then show how to improve this algorithm. We will also report a number of numerical tests to validate our algorithm in Section 4. In addition, we will also show the importance of the exact integration for the solution space with weak regularity.

2 The Eulerian-Lagrangian method

Let Ω be a domain in $\mathbb{R}^d$ and let $T > 0$. For a function $u : \Omega \times [0, T] \to \mathbb{R}$ we consider