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Error Estimates of the Classical and Improved Two-Grid Methods

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Abstract. In this paper, we obtain the first error estimate in L^2-norm for the classical two-grid method, then design an improved two-grid method by adding one more correction on the coarse space to the classical two-gird method. Furthermore, we also present the error estimates in both L^2-norm and H^1-norm for the improved two-grid method. Especially, the L^2 error estimate of the improved two-grid method is one order higher than that of the classical two-grid. At last, we confirm and illustrate the theoretical result by some numerical experiments.

AMS subject classifications: 65N30, 65B99

Key words: Two-grid methods, error estimate.

1 Introduction

The two-grid methods, firstly, introduced by Xu [12] for nonsymmetric or indefine linear elliptic partial differential equations, have been successfully applied to solve many problems in the last two decades, such as nonlinear elliptic problems [13–15, 17], nonlinear parabolic equations [2, 3], Navier-Stokes problems [5, 7], Maxwell equations [19, 20], eigenvalue problems [6, 8, 18, 21], etc. The main idea of the two-grid method is first to solve the original problem in a coarse mesh space with the mesh size H and then to solve a corresponding symmetric positive definite (SPD) problem in a fine mesh space with

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the mesh size $h \ll H$. However, to our knowledge, there exists no work in the literature, which studies the L^2-norm for the two-grid methods.

Let Ω be a bounded domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. we consider the two-grid methods for solving the following nonsymmetric and/or indefinite linear partial differential equations

$$-\mathrm{div}(\alpha(x)\nabla u(x))+\beta(x)\cdot\nabla u(x)+\gamma(x)u(x)=f(x), \qquad x\in\Omega, \tag{1.1a}$$

$$u(x)=g(x), \qquad x\in\partial\Omega, \tag{1.1b}$$

where $\alpha(x)\in\mathbb{R}^{2\times 2}$ is smooth function on $\overline{\Omega}$ satisfying the elliptic condition, namely, for some positive constant α_0,

$$\xi^T\alpha(x)\xi\geq\alpha_0|\xi|^2, \quad \forall\xi\in\overline{\Omega},$$

both $\beta(x)\in\mathbb{R}^2$ and $\gamma(x)\in\mathbb{R}^1$ are also smooth functions on $\overline{\Omega}$, $f(x)$ and $g(x)$ are given functions.

The classical two-grid methods for solving (1.1a)-(1.1b) (see Algorithm 3.1) are referred to [11,12,16], however, only the error estimates in H^1-norm existed in the previous literature. This paper provides an analysis of the first error estimate in L^2-norm for the classical two-grid method and obtains the H^1 error estimate by using a new proof. Furthermore, we design an improved two-grid method by adding one more correction on the coarse space to the classical two-gird method and provide the corresponding error estimates in both L^2-norm and H^1-norm for the improved two-grid method. More details, we has proved that the L^2 error estimate of the improved two-grid method is one order higher than that of the classical two-grid method, although their H^1 error estimates are of the same order. At last, we present some numerical results to assess and validate the theory developed for the proposed method.

To avoid repeated use of generic but unspecified constants, following [10], we use the notation $a\lesssim b$ meaning that there exists a positive constant C such that $a\leq Cb$. The above generic constants C are independent of the function under consideration, but they may depend on Ω and shape-regularity of the meshes.

The remainder of the paper is organized as follows. In Section 2, we present the model problem and some preliminaries. In Section 3, we obtain the first error estimate in L^2-norm for the classical two-grid method. In Section 4, we design and analyze an improved two-grid method. Finally, we report some numerical experiments in support of the efficiency of the method in Section 4.

2 Model problem and preliminaries

In this section, we will discuss nonsymmetric and/or indefinite linear partial differential equations and the corresponding finite element discretizations.