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Shock Profiles for the Shallow-Water Exner Models

C. Berthon[1], B. Boutin[2] and R. Turpault[1,3,*]

[1] *Université de Nantes, Laboratoire de Mathématiques Jean Leray, 2 rue de la Houssinière 44322 Nantes, France*
[2] *Université de Rennes 1, Institut de Recherche Mathématiques de Rennes, Rennes, France*
[3] *Bordeaux-INP, Institut de Mathématiques de Bordeaux, 351 Cours de la Libération 33400 Talence, France*

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Abstract. This article is devoted to analyze some ambiguities coming from a class of sediment transport models. The models under consideration are governed by the coupling between the shallow-water and the Exner equations. Since the PDE system turns out to be an hyperbolic system in non conservative form, ambiguities may occur as soon as the solution contains shock waves. To enforce a unique definition of the discontinuous solutions, we adopt the path-theory introduced by Dal Maso, LeFLoch and Murat [18]. According to the path choices, we exhibit several shock definitions and we prove that a shock with a constant propagation speed and a given left state may connect an *arbitrary* right state. As a consequence, additional assumptions (coming from physical considerations or other arguments) must be chosen to enforce a unique definition. Moreover, we show that numerical ambiguities may still exist even when a path is chosen to select the system's solution.

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1 Introduction

The numerical simulation of sediment transport is essential in many applications. Indeed, a river flow may carry wide volumes of gravels that seriously modify the river bed. As a consequence, the impact of the sediments transport often cannot be neglected when simulating river flows. For instance, water intakes of some industrial installations may be disturbed by bed river modifications or sediment depositions. Recent sediment

*Corresponding author.
Email: rodolphe.turpault@univ-nantes.fr (R. Turpault)

transport tools have been derived to perform numerical simulations for bedload. In general, they are based on a suitable coupling between a solid phase model which governs the evolution of the river bed and a shallow-water model to describe the river flow.

In the present work, we adopt the Exner model [21] to approximate the solid phase. The Exner equation is derived by considering the mass conservation of the solid in the interaction with the river flow. Neglecting dynamical effects, the Exner equation reads:

$$\partial_t z + \partial_x \mathcal{Q}(h,u) = 0, \tag{1.1}$$

where $h>0$ is the water height, u is the height-averaged water speed and z is the height of the river bed. Here, the empirical bedload function $\mathcal{Q}$ is a function of h and u, which is related to the friction between the water and the sediment that forms the river bed. In practice, many forms of $\mathcal{Q}$ are used depending on the physical setup of the considered problem. The reader is referred to [7,19,20,30] where several bedload formulas are detailed. In this paper, we adopt two distinct bedload functions which are quite representative of the different forms of $\mathcal{Q}$ generally considered for physical simulations. The first one is the simple Grass law [24]:

$$\mathcal{Q}(h,u) = \varepsilon u^3, \tag{1.2}$$

and the other one, based on the computation of bed stress, was proposed by Nielsen [31]:

$$\mathcal{Q}(h,u) = \frac{\varepsilon u}{h^{\frac{1}{6}}} \left(\frac{u^2}{h^{\frac{1}{3}}} - \tau_c \right)_+. \tag{1.3}$$

In both cases, ε and τ_c are (usually small) positive parameters and $x_+ = \max(0,x)$.

The full Exner model is then obtained by coupling the Exner equation (1.1) to the shallow-water model for taking into account the topography's variations. This full Exner model therefore reads as follows:

$$\partial_t h + \partial_x hu = 0, \tag{1.4a}$$

$$\partial_t hu + \partial_x \left(hu^2 + g\frac{h^2}{2} \right) + gh\partial_x z = 0, \tag{1.4b}$$

$$\partial_t z + \partial_x \mathcal{Q}(h,u) = 0, \tag{1.4c}$$

where $g>0$ denotes the gravity constant.

To shorten the notations, let us rewrite the system (1.4a)-(1.4b)-(1.4c) in the following condensed form:

$$\partial_t W + \partial_x F(W) + G(W)\partial_x W = 0,$$

where we have set:

$$W = \begin{pmatrix} h \\ hu \\ z \end{pmatrix}, \quad F(W) = \begin{pmatrix} hu \\ hu^2 + g\frac{h^2}{2} \\ \mathcal{Q}(h,u) \end{pmatrix}, \quad G(W) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & gh \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.5}$$