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Algebraic Multigrid Preconditioning for Finite Element Solution of Inhomogeneous Elastic Inclusion Problems in Articular Cartilage

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Abstract. In studying biomechanical deformation in articular cartilage, the presence of cells (chondrocytes) necessitates the consideration of inhomogeneous elasticity problems in which cells are idealized as soft inclusions within a stiff extracellular matrix. An analytical solution of a soft inclusion problem is derived and used to evaluate iterative numerical solutions of the associated linear algebraic system based on discretization via the finite element method, and use of an iterative conjugate gradient method with algebraic multigrid preconditioning (AMG-PCG). Accuracy and efficiency of the AMG-PCG algorithm is compared to two other conjugate gradient algorithms with diagonal preconditioning (DS-PCG) or a modified incomplete LU decomposition (Euclid-PCG) based on comparison to the analytical solution. While all three algorithms are shown to be accurate, the AMG-PCG algorithm is demonstrated to provide significant savings in CPU time as the number of nodal unknowns is increased. In contrast to the other two algorithms, the AMG-PCG algorithm also exhibits little sensitivity of CPU time and number of iterations to variations in material properties that are known to significantly affect model variables. Results demonstrate the benefits of algebraic multigrid preconditioners for the iterative solution of assembled linear systems based on finite element modeling of soft elastic inclusion problems and may be particularly advantageous for large scale problems with many nodal unknowns.

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Key words: Biomechanics, interface problem, conjugate gradient method.

1 Introduction

Biomechanical deformation of articular cartilage, the primary load-bearing soft tissue

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in joints such as the knee, shoulder and hip, is commonly modeled via biphasic continuum mixture theories [16] that idealize the tissue as a fluid-saturated porous medium. In compressive loading at mechanical equilibrium, biphasic deformation of articular cartilage can be modeled based on elasticity theory. However, the presence of a single, sparsely distributed, population of cells (chondrocytes) necessitates the consideration of inhomogeneous inclusion problems in which each cell is idealized as a soft inclusion within a stiff extracellular matrix. Simulating inhomogeneous deformation in the biomechanical microenvironment of the cells in articular cartilage is challenging due to coupled effects among these distinct cartilage regions as they also span disparate length scales (μm to mm), and a wide range of elastic stiffness (KPa to Mpa).

To date, several numerical methods have been developed to study interface problems that exhibit inhomogeneous elastic deformation. For example, in [15], an axisymmetric boundary element method (BEM) for linear elastic domains with internal interfaces was developed. Using direct methods to solve the associated linear algebraic system, the BEM was used to determine linear elastic properties of a pericellular matrix surrounding individual cells via inverse analysis of previously reported experimental data for in situ cell deformation within a cylindrical cartilage explant under static compression. Z. Li and co-authors [6, 7] developed a new immersed interface finite element method to capture the jump conditions along an internal interface for structured meshes. Due to their versatility in generation of unstructured meshes (e.g., via triangular or tetrahedral elements), finite element methods are most commonly used to model elastic deformation in the presence of curved internal interfaces. Use of iterative methods, such as Krylov subspace methods [19], for solution of the assembled linear algebraic systems ensures scalability to problems of moderate to large scale. However, it is well known that the convergence rate of an iterative method depends strongly on the spectral properties of associated operators and, as such, accuracy and efficiency of the associated numerical solutions depend on the choice of algorithm.

Multigrid (MG) methods can be used to significantly accelerate the convergence of iterative methods [2, 20]. When they are well-suited to an application, MG methods exhibit convergence that is independent of problem size [9]. Success of MG techniques is rooted in the differing convergence rates of errors on coarse versus fine grids that are captured by "V-cycles" that traverse the coarse and fine grids during the iterative solution procedure. While initially considered for classical scalar elliptic PDEs, MG methods were later extended to systems of PDEs. Whereas geometric multigrid (GMG) methods require the use of structured hierarchical meshes, algebraic multigrid (AMG) methods effectively induce coarse discretization via direct indexing within the linear algebraic system. Some AMG studies of linear elasticity on unstructured grids relevant to the current work are those of Griebel et al. [8] and Xiao et al. [17], and both studies iterate over multiple V-cycles. In [8], equations of 2D and 3D linear isotropic elasticity were considered and analysis of a blockwise generalization of an AMG method [18] was performed, demonstrating convergence rates independent of problem size. In [17], Xiao et al. considered 2D elastic domains with highly dis-