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Hydrodynamic Regimes, Knudsen Layer, Numerical Schemes: Definition of Boundary Fluxes

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Abstract. We propose a numerical solution to incorporate in the simulation of a system of conservation laws boundary conditions that come from a microscopic modeling in the small mean free path regime. The typical example we discuss is the derivation of the Euler system from the BGK equation. The boundary condition relies on the analysis of boundary layers formation that accounts from the fact that the incoming kinetic flux might be far from the thermodynamic equilibrium.

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1 Introduction

A statistical picture of a cloud of particles leads to the following PDE

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\tau} Q(F), \tag{1.1}$$

satisfied by the particles distribution function

$$F(t, x, v) \geq 0.$$

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Here, N being a positive integer,

$$t \geq 0, \quad x \in \Omega \subset \mathbb{R}^N, \quad \text{and} \quad v \in \mathbb{R}^N,$$

are the time, space and velocity variables respectively. The left hand side of the equation describes the transport of particles, that is the simple motion on straight line with velocity v, and the interactions process the particles are subject to is embodied in the right hand side $Q(F)$, for instance interparticles collisions. The parameter $\tau>0$ is related to the mean free path which is the average distance travelled by the particles without being affected by any interaction †. As τ becomes small, the distribution F tends to an equilibrium F_{eq} that is a function which makes the collision operator vanish

$$Q(F_{\text{eq}}) = 0.$$

By considering conservation laws associated to the collision dynamics, we can then derive macroscopic equations satisfied by moments with respect to the velocity variable v. A difficulty arises when the boundary conditions are not compatible with the equilibrium state. In such a case boundary layers appear, whose analysis is quite delicate. The questions we address are related to the numerical treatment of the boundary layer: considering a system of conservation laws obtained as a small mean free path limit of a kinetic model, what are the associated boundary conditions for the hydrodynamic fields? How the boundary fluxes can be evaluated in numerical procedures?

The paper is organized as follows. First we need to set up a few definition and notation. Our framework will be the Euler system, obtained as the limit of the BGK equation (but the method can be extended to more intricate collision operators like the Boltzmann operator or the Landau-Fokker-Planck operator). The necessary material is recalled in Section 2. In particular entropy dissipation has a central role. A difficulty for hyperbolic equations set on a bounded domain relies on the fact that the number of necessary boundary conditions for the problem to be well-posed usually depends on the solution itself. The entropy provides a natural way to determine the incoming fluxes, with a direct analogy with the microscopic picture. The discussion is strongly inspired by the analysis of the Knudsen layer for the linearized Boltzmann equation by F. Coron-F. Golse-C. Sulem [20], see also the lecture notes of F. Golse [32], and their result is the cornerstone of the definition of numerical fluxes we propose in this paper. Section 3 is the main part of the paper: we discuss how we can take into account the boundary layer in a Finite Volume scheme for the conservation laws. The point is precisely to define a suitable numerical flux on the boundary cells. The definition that we design relies on a decomposition of the numerical solution, according to the nature of the flow (sub or supersonic) combined with an approximation of the half-space problem which defines the matching condition. Finally Section 4 comments the numerical experiments with comparison to direct simulations of the kinetic model.

†Precisely, throughout the paper we shall implicitly work with dimensionless equations so that τ is actually the Knudsen number that is the ratio of a typical (macroscopic) length scale of the flow over the mean free path.