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An Immersed Finite Element Method for the Elasticity Problems with Displacement Jump

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Abstract. In this paper, we propose a finite element method for the elasticity problems which have displacement discontinuity along the material interface using uniform grids. We modify the immersed finite element method introduced recently for the computation of interface problems having homogeneous jumps [20,22]. Since the interface is allowed to cut through the element, we modify the standard Crouzeix-Raviart basis functions so that along the interface, the normal stress is continuous and the jump of the displacement vector is proportional to the normal stress. We construct the broken piecewise linear basis functions which are uniquely determined by these conditions. The unknowns are only associated with the edges of element, except the intersection points. Thus our scheme has fewer degrees of freedom than most of the XFEM type of methods in the existing literature [1,8,13]. Finally, we present numerical results which show optimal orders of convergence rates.

AMS subject classifications: 65N30, 74S05, 74B05

Key words: Elasticity problems, finite element method, Crouzeix-Raviart element, displacement discontinuity.

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1 Introduction

Discontinuities often occur in many model problems in mechanics. For example, they appear along defects of devices, structures, etc. The defects (of material) may arise by pores, cracks, and inclusions. Other kinds of discontinuities can occur between two different solids which interact across a common interface. Examples of this kind are adhesive joints, frictional contacts, laminated structures, composite materials and so on. A simple example happens when two materials of distinct mechanical properties are bonded. Fast and accurate numerical methods to compute the displacements/stresses of such problems have been a challenging task.

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There are several numerical approaches to solve linear elasticity problems by finite element methods (FEM) (see [3,7,36]). For problems with an interface, there are a few methods available. These methods may include: adaptive finite element methods (see [11, 12]), mixed finite element methods (see [30]), and discontinuous Galerkin methods (DG) (see [15, 24, 31]). All of these methods use grids aligned with the interface, which naturally induces unstructured meshes. Hence the structure of the stiffness matrix is complex, and it is difficult to design efficient solver such as multigrid method. This problem becomes more severe if one has to solve time dependent problems in which the interface may move, since it requires regeneration of the mesh for every time step.

In recent years, there have been some developments to solve interface problems using uniform grids. One approach is to use a uniform grid and conventional basis functions such as P_1/Q_1, and add some enrichment functions to them in a hope to cope the discontinuities. The so called extended finite element method (XFEM) was developed by J. Dolbow and T. Belytschko [8] to solve crack modeling problems. After its introduction, the XFEM was successively applied to solve many problems in solid mechanics, such as holes, crack and inclusions, [2,32,33]. Similar schemes were proposed by Becker et al. [1] and Hansbo et al. [13], where the authors proposed certain combined methods of XFEM and Nitsche's penalty methods for weak/strong discontinuities, respectively.

There is a different approach for solving (scalar) interface problems using uniform grids. The immersed finite element methods (IFEM) which use modified basis functions near the interface were introduced in [6, 21, 25–27] and were shown to be effective. The IFEM using Crouzeix-Raviart P_1 element were studied in [22] together with applications to mixed finite volume method. The IFEMs were used to solve various types of problems such as nonhomogeneous jumps case [5, 10, 18], parabolic equations [16], elasticity equations [17, 20, 28, 29, 34], electrical potential interface problem [4], PIC simulations in ion optics [19] and so on.

Recently, the IFEMs for elasticity problems with interface have been studied by various authors in different contexts. Lin et al. [29] solved planar elasticity problems with homogeneous interface conditions $[\mathbf{u}]_\Gamma=\mathbf{0}$ and $[\sigma(\mathbf{u})\mathbf{n}]_\Gamma=\mathbf{0}$ using IFEM based on linear/bilinear finite element. Hou et al. [17, 34] studied linear based IFEM including nonhomogeneous jumps in which they claim second order convergence in L^∞-norm. But it is well known that such elements suffer locking phenomena for the nearly incompressible case, see [9] for example. In [28], Lin et al. used Rannacher Turek element on rectangular grid for solving elasticity problems with homogeneous interface conditions. The numerical results shows optimal order convergence in H^1 and L^2 norms, but no theory was provided.

Kwak et al. [20] proposed an IFEM based on Crouziex-Raviart P_1 element with stability term to solve elasticity problems with homogeneous interface conditions, where they provided the convergence proof and the optimal numerical results in L^2 and H^1-norms. One of the differences from the above IFEMs is the addition of stability terms $\int_e \frac{\tau}{h}[\mathbf{u}][\mathbf{v}]ds$. It is well-known that the lowest order linear/bilinear element are not stable since the associated bilinear forms are not coercive [9, 14]. Brenner and Sung [3] inves-