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On the Full C_1-Q_k Finite Element Spaces on Rectangles and Cuboids

Shangyou Zhang*

[1] *Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716-2553, USA*

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Abstract. We study the extensions of the Bogner-Fox-Schmit element to the whole family of Q_k continuously differentiable finite elements on rectangular grids, for all $k \geq 3$, in 2D and 3D. We show that the newly defined C_1 spaces are maximal in the sense that they contain all C_1-Q_k functions of piecewise polynomials. We give examples of other extensions of C_1-Q_k elements. The result is consistent with the Strang's conjecture (restricted to the quadrilateral grids in 2D and 3D). Some numerical results are provided on the family of C_1 elements solving the biharmonic equation.

AMS subject classifications: 65M60, 65N30

Key words: Differentiable finite element, biharmonic equation, Bogner-Fox-Schmit rectangle, quadrilateral element, hexahedral element, Strang's conjecture.

1 Introduction

It is relatively difficult to construct continuously-differentiable finite elements in two and three space dimensions. Most such C_1 elements were designed in 1970s and earlier (cf. Ciarlet [10]). Most C_1 elements were constructed on triangles and tetrahedra with piecewise polynomials P_k. As usual, P_k and Q_k stand for polynomials of total degree and separate degree k or less, respectively. For example, we have the Argyris P_5-triangle (1968), the Bell reduced P_5-triangle (1969), the Morgan-Scott P_k-triangles ($k \geq 5$) (1975), the Hsieh-Clough-Tocher P_3-macrotriangles (1965), the reduced Hsieh-Clough-Tocher P_3-macrotriangles (1976), the Douglass-Dupont-Percell-Scott P_k-triangles (1979), the Powell-Sabin P_2-triangles (1977), the Fraeijs de Veubeke-Sander P_3 quadrilateral and its reduced version (1964), cf. [2,4,10–12,14,15,22,25,27,28,37]. The

*Corresponding author.
URL: `http://www.math.udel.edu/~szhang/`
Email: szhang@udel.edu (S. Y. Zhang)

last two elements carry the term quadrilateral in the name, but they are P_k macrotriangle elements too. It seems that the Bogner-Fox-Schmit rectangle (1965) is the only C_1 element on rectangular grids, cf. [9,10]. Nevertheless, there are still quite some work on this one of the oldest elements, mainly due to its simplicity and effectiveness in computation, cf. [1,20,24,26].

In this paper, we study the Bogner-Fox-Schmit element extended to higher order Q_k elements in 2D and 3D, $k\geq3$. Such extensions were done also in [8,13,21]. There might not be much interest in application to use high order elements, though they provide usually a better accuracy with less number of unknowns. For example, as shown in our numerical tests, the Q_4 element performs better than its Q_3 cousin, the Bogner-Fox-Schmit element. However, our main interest in studying C_1-Q_k elements is to understand the structure and approximation property of C_0-Q_{k-1} element under the divergence-free or the nearly-incompressible constraint, cf. two subsequent researches [38,39]. The approach is standard. Morgan and Scott [22] modified Argyris P_5-triangles to cover all C_1-P_5 functions on triangular grids, and extended it to C_1-P_k for all $k\geq5$. Scott and Vogelius [29,30] showed that C_0-P_k elements for all $k\geq4$ provide the optimal-order approximation property on general triangular grids under the incompressibility constraint, for fluids and elasticity. The generalization of Scott and Vogelius work to Q_k polynomials is not accomplished yet. There are some work on Q_k elements under the incompressibility constraint and the element is shown suboptimal, cf. [3,32].

The construction of high-order C_1 finite elements is relatively easy, compared with that of low-order elements. Such a construction consists of two parts, the local uniqueness and polynomial preserving, and the global inter-element coupling. We note that Gopalacharyulu made an extension to the Bogner-Fox-Schmit element in [17]. The extension is not a higher order element, but an element which includes some higher order polynomial terms so that the element may work better for plates. Our work here extends the element of Gopalacharyulu, so that the higher order approximation can be guaranteed. In fact, it was pointed out by Watkins that the construction of Gopalacharyulu missed some lower order terms while adding higher order terms to the Bogner-Fox-Schmit element, cf. [36]. To correct it, Gopalacharyulu added some more terms into the element, however, without showing the extension is conforming (C_1), neither complete, in [18]. For the extensions studied in this paper, we show their completeness (the optimal order of approximation), fullness (including all C_1-Q_k polynomials), and conformity. This is mainly the work further that of [8,13,21].

For C_1 piecewise polynomials on triangular grids, Strang gave a conjecture on the dimension based on the inter-element constraint, cf. [5,23,33,34]. The conditions and validity of the Strang's conjecture are open problems, cf. [23]. But we will show the conjecture holds on rectangular grids, both in 2D and 3D.

The paper has three additional sections. In Section 2, 2D C_1-Q_k elements are constructed for all $k\geq3$. In Section 3, 3D C_1-Q_k elements are constructed for all $k\geq3$. In Section 4, a simple numerical test on the biharmonic equation is performed with the Bogner-Fox-Schmit element and higher order C_1-Q_k elements.