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A Revisit on the Derivation of the Particular Solution for the Differential Operator $\Delta^2 \pm \lambda^2$

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Abstract. In this paper, we applied the polyharmonic splines as the basis functions to derive particular solutions for the differential operator $\Delta^2 \pm \lambda^2$. Similar to the derivation of fundamental solutions, it is non-trivial to derive particular solutions for higher order differential operators. In this paper, we provide a simple algebraic factorization approach to derive particular solutions for these types of differential operators in 2D and 3D. The main focus of this paper is its simplicity in the sense that minimal mathematical background is required for numerically solving higher order partial differential equations such as thin plate vibration. Three numerical examples in both 2D and 3D are given to validate particular solutions we derived.

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1 Introduction

The idea of splitting a given partial differential equation into solving a homogeneous equation and an inhomogeneous equation is well known. In recently years, such approach becomes very popular for various boundary meshless methods such as the Trefftz method, the method of fundamental solutions [8, 9], and the boundary knot method (BKM) [4], etc. By evaluating the particular solution, these boundary meshless methods can be extended from solving only homogeneous equations to inhomogeneous equations and time-dependent problems [1, 9]. As a consequence, many

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numerical techniques have been developed to evaluate particular solutions for various types of partial differential equations. Chen and Rashed [2] were first to extend the derivation of particular solutions to Helmholtz-type equations using thin plate splines. Muleshkov et al. [11] further extended the concept to polyharmonic splines. However, the derivation of particular solutions using the annihilator method and algebraic techniques in [11] were too tedious to use for solving complicated differential operators. Cheng [6] revisited the problem using the technique of fundamental solutions so that particular solutions can be easily derived. Recently, Muleshkov and Golberg [12], and Chen et al. [3] derived particular solution for more complicate differential operators using radial basis functions and Chebyshev polynomials. In 2009, Tsai et al. [13] extended the derivations of particular solutions to polyharmonic, poly-Helmholtz operators and their products.

In contrast to the tedious derivation of particular solutions for Helmholtz-type differential operators shown in [11] and its extension to general operators [13], we propose a simple algebraic factorization approach to derive particular solutions for the differential operators

$$\Delta^2 \pm \lambda^2,$$

in 2D [5] and 3D using polyharmonic splines. On the other hand, Young et al. [14] solved the homogeneous equation of plate vibration problem in which $\Delta^2-\lambda^2$ is the differential operator. Coupled with the particular solutions derived in this paper, [14] can be effectively extened to solving the arbitrarily loaded flexural vibrations of an uniform thin plate.

This paper is organized as follows. In section 2, we derive particular solutions for polyharmonic splines which includes two dimensional and three dimensional cases. In Section 3, we derive particular solutions for the monomial term for $\Delta^2-\lambda^2$. In Section 4, numerical examples for two 2D examples and one 3D example are given. In Section 5, we conclude this paper with opening issues and future applications.

2 Particular solutions for polyharmonic splines

Let us consider the following boundary value problem

$$\left(\Delta^2-\lambda^2\right)u=f(\mathbf{x}), \qquad \mathbf{x}\in\Omega\subset\mathbb{R}^d, \tag{2.1}$$
$$\mathcal{B}_1 u=g(\mathbf{x}), \qquad \mathbf{x}\in\partial\Omega, \tag{2.2}$$
$$\mathcal{B}_2 u=h(\mathbf{x}), \qquad \mathbf{x}\in\partial\Omega, \tag{2.3}$$

where λ is a non-zero constant, Δ is the Laplacian, $\mathcal{B}_1$ and $\mathcal{B}_2$ are the boundary differential operators, f,g and h are given functions, and Ω is an open bounded domain in $\mathbb{R}^d$, d=2, 3, with boundary $\partial\Omega$. Note that $\mathbf{x}=(x,y)$ in 2D and $\mathbf{x}=(x,y,z)$ in 3D. For d=2, (2.1)–(2.3) govern the loaded flexural vibrations of a uniform thin plate.

Let u_p be a particular solution of the governing equation, then it satisfies

$$\left(\Delta^2-\lambda^2\right)u_p=f(\mathbf{x}), \tag{2.4}$$