Advances in Applied Mathematics and Mechanics June 2009
Adv. Appl. Math. Mech., Vol. **1**, No. 3, pp. 438-450 (2009)

LWDG Method for a Multi-Class Traffic Flow Model on an Inhomogeneous Highway

Tiantian Sun[1] and Jianxian Qiu[1,*]

[1] *Department of Mathematics, Nanjing University, Nanjing, Jiangsu 210093, China*

Received 12 February 2009; Accepted (in revised version) 01 April 2009
Available online 22 April 2009

Abstract. In this paper, we apply the discontinuous Galerkin method with Lax-Wendroff type time discretizations (LWDG) using the weighted essentially non-oscillatory (WENO) limiter to solve a multi-class traffic flow model for an inhomogeneous highway. This model is a kind of hyperbolic conservation law with spatially varying fluxes. The numerical scheme is based on a modified equivalent system which is written as a "standard" hyperbolic conservation form. Numerical experiments for both the Riemann problem and the traffic signal control problem are presented to show the effectiveness of the method.

AMS subject classifications: 35L65, 65M60

Key words: Multi-class traffic flow; inhomogeneous highway; spatially varying flux; non-strictly hyperbolic conservation laws; LWDG method; WENO limiter.

1 Introduction

The first discontinuous Galerkin (DG) method was introduced in 1973 by Reed and Hill [18], in the framework of neutron transport (steady state linear hyperbolic equations). A major development of the DG method was carried out by Cockburn et al. in a series of papers [2–6], in which a framework was established to solve *nonlinear* hyperbolic conservation laws:

$$\begin{cases} u_t + \nabla \cdot f(u) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

They proposed to use an explicit, nonlinear stable and high order Runge-Kutta time discretizations [19] and DG discretization in space with exact or approximate Riemann solvers as interface flux and limiters such as the total variation bounded (TVB) limiters [20] or weighted essential non-oscillatory (WENO) type limiters [15, 16] to

*Corresponding author.
URL: `http://math.nju.edu.cn/~jxqiu/index.htm`
Email: asmaphy84@yahoo.com.cn (T. Sun), jxqiu@nju.edu.cn (J. Qiu)

achieve nonoscillatory properties for strong shocks. The method is termed as Runge-Kutta discontinuous Galerkin method (RKDG). The DG method has following advantages: Easy handling of complicated geometry and boundary conditions (common to all finite element methods), allowing hanging nodes in the mesh; Compact, communication only with immediate neighbors, regardless of the order of the scheme; Explicit, because of the discontinuous basis, the mass matrix is local to the cell, resulting in explicit time stepping (no systems to solve); Parallel efficiency, achieves 99% parallel efficiency for static mesh and over 80% parallel efficiency for dynamic load balancing with adaptive meshes [1].

An alternative approach to discretize the time derivative term could be using a Lax-Wendroff type time discretization procedure, which is also called the Taylor type referring to a Taylor expansion in time. This approach is based on the idea of the classical Lax-Wendroff scheme [11], and it relies on converting all the time derivatives in a temporal Taylor expansion into spatial derivatives by repeatedly using the PDE and its differentiated versions. The spatial derivatives are then discretized by the DG approximations. The Lax-Wendroff type time discretization, which is also referred to as the Taylor-Galerkin method for the finite element methods, usually produces the same high order accuracy with a smaller effective stencil than that of the Runge-Kutta time discretization, and it uses more extensively the original PDE. Since the Lax-Wendroff time discretization is an one step method instead of the multi-step Runge-Kutta time discretization, the LWDG method can save a certain amount of computational cost over the RKDG method, thus is more cost effective.

Lighthill and Whitham [13] and Richards [17] independently proposed a simple continuum model, known as the LWR model, to describe the characteristics of traffic flow. In this model, a traffic stream model (relationship between traffic state variables of flow, speed and density, e.g., [10]) is supplemented by the continuity equation of vehicles, and the resulting partial differential equation presumably could be solved to obtain the density as a function of space and time. Although aiming at providing a coarse representation of traffic behavior, the LWR model is capable of reproducing qualitatively a remarkable amount of real traffic phenomena such as shock formation. However, there are still some puzzling traffic phenomena that this simple LWR model cannot address or explain, such as the two-capacity or reverse-λ state in the fundamental diagram, hysteresis of traffic flow and platoon dispersion.

Recently, multi-class models (MCLWR models) have been developed in an attempt to explain these puzzling traffic phenomena by modeling users' lane changing behavior and multiple vehicle types [7, 8]. Although the MCLWR model is simple in nature, it was found that the model is capable of producing the desired properties of a macroscopic traffic flow model and it explains many puzzling phenomena mentioned before. In [23], the MCLWR model was solved by a first-order Lax-Friedrichs finite difference scheme. However, this scheme may produce smeared solutions near discontinuities due to excessive numerical viscosity. Then Lebacque [12] successfully applied the Godunov scheme, introduced by Godunov [9], to solve the LWR model. It is subject to smaller numerical viscosity, but requires a Riemann solver as its building block,