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Numerical Validation of Brenner's Hydrodynamic Model by Force Driven Poiseuille Flow

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Abstract. Recently Brenner [Physica A **349**, 60 (2005)] proposed a modified Navier-Stokes set of equations. Based on some theoretical arguments and some limited experiments, the model is expected to be able to describe flows with a finite Knudsen number. In this work, we apply this model to the plane Poiseuille flow driven by a force, and compare the results with the Direct Simulation Monte Carlo (DSMC) measurements. It is found that Brenner's model is inadequate for flows with a finite Knudsen number.

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1 Introduction

The Navier-Stokes (NS) set of equations is a sound and robust theoretical model for continuum fluid dynamics where the local thermodynamic equilibrium assumption holds. Although the NS model has gained much successes in many applications, it encounters some challenging difficulties for non-continuum flows which exhibit a finite Knudsen number Kn defined as Kn$=\lambda/H$, where λ is the mean-free-path of the gas and H is a characteristic length of the flow. Non-continuum flows have been widely studied in the rarefied gas community, where the gas density is usually very low so that the mean-free-path of the gas is relatively large. In recent years, non-continuum flows with a normal gas density but with a small characteristic length have also attracted much attention with the rapid development in microelectromechanical systems. Due to the finite Knudsen number effect, the continuum-equilibrium assumption may break down and the NS model will fail to work for these flows [1, 2].

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Some hydrodynamic models beyond the NS equations, such as the Burnett equations, super-Burnett equations, and 13-moments equations, have been proposed from different viewpoints [3–10].

These extended hydrodynamic models are usually derived from gas kinetic theory (the Boltzmann equation) [11]. Unfortunately, these extended continuum models are exposed to some criticisms [12], such as the validity of the Chapman-Enskog expansion for large Kn, the difficulties in ascertaining the boundary conditions, and the inherent instabilities. What is more disappointing is that most of these higher-order models cannot even describe the simple Kramer's problem correctly [13].

Owing to the difficulties arising in the non-Navier-Stokes hydrodynamic models, there are increasing interests to rescue the NS model for non-continuum flows in recent years [14–20]. These models are still in the NS framework, and share most of the advantages of the NS model such as the simple structure and the easy implementation. Recently, Brenner proposed one such model based on a new picture of the fluid velocity [21–23], which we will term as "Brenner-Navier-Stokes" (BNS) model in this work. Compared with other extended hydrodynamics models derived from the Boltzmann equation, the BNS is much simpler because only one single additional term is introduced into each of the momentum and energy equations. Furthermore, although the BNS model was derived phenomenologically, there are some independent experimental and theoretical evidences that it has the potential to model non-continuum flows [21,22,24–26].

However, as a new hydrodynamic model the BNS equations should be tested by some well-accepted benchmark problems before acceptance. In this work, we will investigate the applicability of the BNS model for rarefied gas flows with a small but finite Knudsen number (Kn$\leq$0.1). The test problem employed here is a force driven Poiseuille flow between two parallel plates. Both Direct Simulation Monte Carlo (DSMC) and kinetic theory have shown that even with a small Kn, the pressure and temperature profiles in this flow exhibit qualitatively different behaviors from those predicted by the NS equations [27–31]. Therefore, this flow can serve as a good test problem for any extended hydrodynamic models intended for non-continuum flows.

2 The Brenner-Navier-Stokes equations

Generally, the hydrodynamic equations that govern the fluid motion can be expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}_m) = 0, \tag{2.1}$$

$$\frac{\partial (\rho \boldsymbol{u}_m)}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}_m \boldsymbol{u}_m) - \nabla \cdot \boldsymbol{P} = \rho \boldsymbol{a}, \tag{2.2}$$

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}_m e) + \nabla \cdot \boldsymbol{j}_e - \nabla \cdot (\boldsymbol{P} \cdot \boldsymbol{u}_m) = \rho \boldsymbol{a} \cdot \boldsymbol{u}_m, \tag{2.3}$$