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Exact and Approximate Values of the Period for a "Truly Nonlinear" Oscillator: $\ddot{x} + x + x^{1/3} = 0$

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Abstract. We investigate the mathematical properties of a "truly nonlinear" oscillator differential equation. In particular, using phase-space methods, it is shown that all solutions are periodic and the fixed-point is a nonlinear center. We calculate both exact and approximate analytical expressions for the period, where the exact solution is given in terms of elliptic functions and the method of harmonic balance is used to calculate the approximate formula.

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1 Introduction

For the past several centuries the study of nonlinear oscillations has centered on systems for which the elastic force terms have a harmonic oscillator limiting form [1,3,8]. Thus, the corresponding mathematical models have the structure

$$\ddot{x} + \omega^2 x = \epsilon f(x, \dot{x}), \tag{1.1}$$

where ϵ is a parameter which can be taken to be non-negative; the dot over x indicates a time derivative, i.e., $\dot{x} \equiv dx/dt$ and $\ddot{x} \equiv d^2x/dt^2$; and $f(x, \dot{x})$ is, generally, a nonlinear function of its arguments. In the limit $\epsilon \to 0$, Eq. (1.1) becomes

$$\ddot{x} + \omega^2 x = 0, \tag{1.2}$$

which is the equation of motion for the linear harmonic oscillator. If the parameter ϵ is small, i.e., $0 < \epsilon \ll 1$, then a number of techniques [1, 3] can be applied to construct/calculate analytic approximations to the oscillatory solutions of Eq. (1.1). However, it should be noted that all such standard procedures are based on the assumption

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that the limiting equation Eq. (1.2) can be obtained from Eq. (1.1). From this it follows that the required oscillatory solutions can be represented by means of generalized asymptotic expansions in ϵ [1].

In recent years, a new class of nonlinear oscillatory differential equations have been studied [6,7]. These equations do not have a harmonic oscillator limiting form when the parameter ϵ goes to zero. Specific examples include the following:

$$\ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}, \tag{1.3a}$$

$$\ddot{x} + |x|x = -\epsilon\dot{x}, \tag{1.3b}$$

$$\ddot{x} + x + x^{1/3} = -\epsilon x^2\dot{x}. \tag{1.3c}$$

Observe that Eqs. (1.3a) and (1.3b) have elastic force terms given respectively, by $-x^3$ and $-|x|x$, and therefore they have no linear terms. However, the elastic force in Eq. (1.3c) is $-x - x^{1/3}$, and for small x is dominated by $x^{1/3}$ rather than x; consequently, for sufficiently small x, its equation of motion can be approximated by

$$\ddot{x} + x^{1/3} \approx 0. \tag{1.4}$$

Thus, none of these three equations have the proper harmonic oscillator limiting form required for the application of the standard perturbation methods [1,3].

All of these examples are specific cases of the general equation

$$\ddot{x} + g(x) = \epsilon f(x, \dot{x}), \tag{1.5}$$

where $f(x, \dot{x})$ is a polynomial-type function of x and $\dot{x}$, and $g(x)$ the elastic force term, does not have a linear approximation for $x \to 0$, i.e.,

$$g(x) = \mathcal{O}(x^\alpha), \qquad \alpha \neq 1. \tag{1.6}$$

We denote such equations "truly nonlinear oscillatory" (TNL) differential equations, and in several publications have presented techniques to calculate analytic approximations to any periodic solutions they may possess [6,7].

The main purpose of this paper is to investigate the general properties of the following TNL differential equation

$$\ddot{x} + x + x^{1/3} = 0, \tag{1.7}$$

with the main effort placed on calculating both exact and approximate expressions for the period of oscillations. In the next section, we give some preliminaries relating to the fundamental properties of the solutions to Eq. (1.7). Section 3 gives our calculation for the exact period, while in Section 4, an approximate formula for the period is derived. Finally, in the last section, we summarize our results and indicate the next steps in the analysis of Eq. (1.7).