Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **11**, No. 5, pp. 1114-1135

DOI: 10.4208/aamm.OA-2018-0221
October 2019

A New Fifth-Order Trigonometric WENO Scheme for Hyperbolic Conservation Laws and Highly Oscillatory Problems

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Received 18 October 2018; Accepted (in revised version) 9 March 2019

Abstract. In this paper, we propose trigonometric polynomial reconstructions based on one five-point stencil and two two-point stencils, instead of algebraic polynomial reconstructions defined on three three-point stencils [20, 35], as a building block for designing a fifth-order trigonometric weighted essentially non-oscillatory (TWENO) scheme to solve hyperbolic conservation laws and highly oscillatory problems. The main objective of the paper is to extremely reduce the difficulty in computing the linear weights, could get less absolute truncation errors in smooth regions, and keep sharp shock transitions in nonsmooth regions. Extensive benchmark numerical tests including some highly oscillatory problems are provided to verify the good performance of the new scheme.

AMS subject classifications: 65M60, 35L65

Key words: Trigonometric polynomial reconstruction, hyperbolic conservation laws, highly oscillatory problem, WENO scheme.

1 Introduction

In this paper, we design new trigonometric polynomial reconstructions as building blocks for designing a fifth-order trigonometric weighted essentially non-oscillatory (TWENO) scheme to solve hyperbolic conservation laws

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\begin{cases} u_t + \nabla \cdot f(u) = 0, \\ u(x_1, \cdots, x_n, 0) = u_0(x_1, \cdots, x_n). \end{cases} \tag{1.1}
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There are two major advantages of the new finite difference trigonometric WENO scheme superior to the classical WENO schemes [20,35]: the first is the associated linear weights can be set as any positive parameters on condition that their sum is one, and the second is its simplicity and easy extension to multi-dimensions in engineering applications for highly oscillatory problems.

On the one hand, many numerical methods were designed to improve the first-order methods [14] to arbitrary numerical order over the past years. For the purpose of achieving uniform high-order accuracy, Harten and Osher gave a weaker version of the TVD (Total Variation Diminishing) criterion [17] and then proposed the essentially non-oscillatory (ENO) type schemes [18]. In 1994, Liu et al. [24] proposed a third-order finite volume weighted ENO (WENO) scheme on structured meshes. In 1996, Jiang and Shu [20, 35] proposed a new framework to construct finite difference WENO schemes from the r-th order to get $(2r-1)$-th order accuracy, gave a new way of measuring the smoothness indicators, and emulated the ideas of minimizing the total variation of the approximation. Many other classical WENO schemes were developed and applied, such as the central/compact WENO (CWENO) schemes [5, 8, 11, 22, 23, 29, 30, 34], two-dimensional finite volume WENO schemes on triangular meshes [19], three-dimensional finite volume WENO on structured meshes [37] and tetrahedral meshes [40], WENO and Hermite WENO (HWENO) schemes and as limiters for solving Runge-Kutta discontinuous Galerkin (RKDG) methods [25,31–33], and so on. More recently, a new class of fifth-order WENO schemes has been designed to solve hyperbolic conservation laws [44,45]. For related work we refer to [1, 5, 9, 13, 21–23, 34]. A key idea of such various WENO schemes is the application of a linear combination of lower order fluxes or reconstruction to obtain a higher order approximation. Both ENO and WENO schemes use the idea of adaptive spatial stencils to automatically obtain high-order accuracy in smooth regions and keep non-oscillatory property in nonsmooth regions. All ENO and WENO schemes have been quite successful in numerical simulations, such as some benchmark problems containing strong shocks, contact discontinuities, and complicated smooth solution structures.

On the other hand, numerical methods based on the trigonometric polynomial functions appear to be very suitable for the simulation of some highly oscillatory problems or wave-like phenomena. Although the algebraic polynomial reconstruction is a good building block to design a numerical flux, it cannot be modified according to the characteristic of the given highly oscillatory data. When interpolating highly oscillatory data, a numerical scheme based on the trigonometric polynomial reconstruction instead of the algebraic polynomial reconstruction would be more suitable for solving such highly oscillatory problems.

In this research direction, it is noted that there have been quite long list of literatures. Let us review the developing history of such trigonometric polynomial reconstructions